PROJECTIONS OF ORBITS AND ASYMPTOTIC BEHAVIOUR OF MULTIPLICITIES FOR COMPACT LIE GROUPS

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INTRODUCTION

In 1874 the Norwegian mathematician Sophus Lie wrote the paper " Zur Theorie des Integrabilitätsfaktors " [21], in which he treated the following problem :

How can the stability of a differential equation under a group of transformations be used towards its integration ?

The n-parameter transformation groups, which Lie considered, could be described locally by n real parameters and the group operations were smooth in these local coordinates. Nowadays these groups are called Lie groups.

A somewhat simpler object than the Lie group itself is the set of those vector fields on the Lie group, which are invariant under left multiplication. They form an algebra under the commutator product $[X,Y] = XY - YX$, the so-called Lie algebra of the Lie group. The bracket $[.,.]$ is bilinear, anti-symmetric and satisfies the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

A fundamental result of Lie says that a Lie group is locally completely determined by its Lie algebra.

A unitary representation of a Lie group G on a complex Hilbert space H is a continuous homomorphism of G into the group of unitary operators on H. If H' is a closed G-invariant subspace of H, so is its orthogonal complement, so that H splits as a direct sum of two subrepresentations. Two representations of G are said to be equivalent if there exists a linear isomorphism between the representation spaces, which intertwines the action of G.

A representation is called irreducible if the only closed G-invariant subspaces are zero and the whole space. One of the major problems in Lie group theory is :

How to classify all irreducible unitary representations up to equivalence ?

The set of equivalence classes of irreducible unitary representations of a Lie group G is denoted by $G^\wedge$.

The simplest example is the case of a connected abelian Lie group A with Lie algebra $\mathfrak{a}$. Essentially there are two of them, the Euclidean space $\mathbb{R}^n$ and the torus $\mathbb{R}^n/\mathbb{Z}^n$. Each irreducible unitary representation of A is one-dimensional and of the form

$$x \longrightarrow e^{2\pi\sqrt{-1}.(x,y)}$$

with $x,y \in \mathbb{R}^n$ for $A = \mathbb{R}^n$ and $x \in \mathbb{R}^n/\mathbb{Z}^n$, $y \in \mathbb{Z}^n$ for $A = \mathbb{R}^n/\mathbb{Z}^n$. In both cases $A^\wedge$ can be identified in a canonical way with a subset of the real vector space $\sqrt{-1}.\mathfrak{a}^*$. Indeed, for $A = \mathbb{R}^n$ we have $A^\wedge \simeq \sqrt{-1}.\mathfrak{a}^*$ and for $A = \mathbb{R}^n/\mathbb{Z}^n$ we can identify $A^\wedge$ with a lattice in $\sqrt{-1}.\mathfrak{a}^*$, the weight lattice of the torus. If we define for $x \in A$ the operator T_x on $L^2(A)$ by

$$T_x f(y) = f(x+y)$$

then we get a natural representation of A on $L^2(A)$. Decomposing this representation into a (continuous) direct sum of irreducible representations yields the Plancherel formula in the theory of Fourier integrals and Fourier series respectively.

A Lie group acts on itself by conjugation. Differentiation at the identity yields a representation of G on the Lie algebra $\mathfrak{g}$ of G, the so-called adjoint representation. The dual of the adjoint representation is called the coadjoint representation.

Let K be a connected compact Lie group, for example $SU(n,\mathbb{C})$ or $SO(n,\mathbb{R})$. Each irreducible unitary representation π of K is finite

dimensional. The character χ_π of π is a complex valued function on K defined by

$$\chi_\pi(k) = \mathrm{Trace}(\pi(k))$$

Similar to the case of finite groups one can show that any irreducible unitary representation is completely determined up to equivalence by its character. Choose a maximal torus T in K. The dimension of T is an invariant, and called the rank of K. The normalizer W of T in K acts on T by conjugation and on $\mathfrak{t}$ by the adjoint representation as a finite group generated by reflections, the so-called Weyl group of the pair (K,T). An important fact is that each conjugation orbit intersects T transversely in a Weyl group orbit. Being a conjugation invariant function the character of π is completely determined by its restriction to T. We write

$$\chi_\pi|_T = \sum_{\mu \in T^\wedge} m_\pi(\mu).\mu$$

where the non-negative integer $m_\pi(\mu)$ is the multiplicity of μ in the restriction of π to T. For $T^\wedge$ we also write Λ_w, the weight lattice of T. The elements μ of Λ_w, for which $m_\pi(\mu)$ is positive, are called the weights of the representation π. The set of weights of π is denoted by $\mathcal{W}(\pi)$. A complete classification of $K^\wedge$ goes back to É. Cartan in 1913 [5] :

The set of extremal points in $\mathcal{W}(\pi)$ consists of one single Weyl group orbit $W.\lambda$ for some $\lambda \in \Lambda_w$, and π is completely determined by this orbit. Moreover, ech Weyl group orbit in Λ_w occurs in this way.

The set of roots Δ of the pair (K,T) is the set of non-zero weights of the adjoint representation. They all have multiplicity one, so that the number of roots plus the rank of K is equal to the dimension of K. The root lattice Λ_r is the sublattice of Λ_w generated by the roots. Fix a K-invariant inner product on $\sqrt{-1}.\mathfrak{t}^*$. Then the Weyl group is generated by the reflections s_α in the hyperplane V_α perpendicular to $\alpha \in \Delta$. A Weyl chamber is the closure of a connected component of the complement of all V_α's in $\sqrt{-1}.t^*$. Fix a Weyl chamber C^+. A root α is called positive if $(\alpha,\beta) \geq 0$ for all $\beta \in C^+$, and the set of positive roots is denoted by Δ^+. Because C^+ is a fundamental domain for the action of W on $\sqrt{-1}.t^*$, there exists for each $\pi \in K$ a unique $\lambda \in C^+ \cap \Lambda_w$ such that the extremal points of $W(\pi)$ are just the orbit $W.\lambda$. We write $\pi(\lambda,K)$ for π, the so-called representation with highest weight λ.

Let N be a connected nilpotent Lie group with Lie algebra n. In 1962 A.A. Kirillov pointed out that the set $N^\wedge$ could be parametrized by certain orbits of the coadjoint action [18]. The representation corresponding to an orbit in n^* is realized as a certain L^2-function space. The character of the representation which is defined as a distribution on the group N is equal to the normalized invariant measure supported by the corresponding orbit. Let M be a connected closed subgroup of N. By restricting a linear functional $f \in n^*$ to m we get a natural projection p of n^* onto m^*. If an irreducible unitary representation π of N corresponds to the orbit O_π in n^*, then its restriction to M is decomposed into a direct integral of irreducible unitary representations of the subgroup M corresponding to the M-orbits which belong to $p(O_\pi)$. This property is called the functorial property of the orbit method.

It has been remarked by several people that the representation theory of a connected compact Lie group K can also be decomposed in terms of orbits of the coadjoint representation. An orbit in k^* under the coadjoint action is said to be integral for K if the intersection with t^*, which is a Weyl group orbit, is contained in the weight lattice of T (here we identify t^* with $\sqrt{-1}.t^*$). Then $K^\wedge$ is parametrized by the set of integral orbits in k^*. The representation space can be realized as the set of holomorphic sections in a certain line bundle on this orbit. Moreover, the character can be expressed

locally as an orbital integral.

The main problem studied in this thesis is to what extent the functorial property of the orbit method holds in the compact case.

Therefore let L be a connected Lie subgroup of K. We have restricted ourselves to subgroups of the same rank, so we may assume that T is contained in L. We use the subscripts K and L to distinguish between the corresponding objects of K and L respectively. For example, W_K is the Weyl group of the pair (K,T) and W_L the Weyl group of the pair (L,T). For $\lambda \in C_K^+ \cap \Lambda_w$ the restriction of $\pi(\lambda,K)$ to L splits as a direct sum of irreducible representations $\pi(\mu,L)$, $\mu \in C_L^+ \cap \Lambda_w$, with multiplicity $m_\lambda^{K,L}(\mu)$. In order to study the asymptotic behaviour of the integers $m_\lambda^{K,L}(\mu)$ as $|\lambda|$ tends to infinity, we introduce in Chapter 3 a piece-wise polynomial function $M_\lambda^{K,L} : \sqrt{-1}.t^* \to \mathbb{R}$, which satisfies the relation

$$M_{t\lambda}^{K,L}(t\mu) = t^r.M_\lambda^{K,L}(\mu)$$

for $t > 0$ and $r = |\Delta_K^+ \setminus \Delta_L^+| - \text{rank}(\Delta_K^+ \setminus \Delta_L^+)$. We call $M_\lambda^{K,L}$ the asymptotic multiplicity function because of the following theorem.

<u>Theorem 4.</u> There exists a constant $C \in \mathbb{R}^+$, so that for all $\lambda \in C_K^+ \cap \Lambda_w$ and $\mu \in C_L^+ \cap (\lambda + \Lambda_r)$ we have

$$|m_\lambda^{K,L}(\mu) - M_\lambda^{K,L}(\mu)| \leq C.(1 + |\lambda|)^{r-1}$$

$\Lambda_r = \Lambda_{r,K}$

In fact, $M_\lambda^{K,L}$ is a sort of continuous analogue of $m_\lambda^{K,L}$. As a distribution $M_\lambda^{K,L}$ satisfies a differential equation, which is analogous to a difference equation for $m_\lambda^{K,L}$. The multiplicity function $m_\lambda^{K,L}$ is skew-invariant under a certain affine action of W_L. The function $M_\lambda^{K,L}$, however, is skew-invariant with respect to the ordinary action of W_L.

In general one can write K locally as a direct product of a compact semisimple Lie group and a torus. To simplify the notations we shall assume that K is semisimple. Let the Euclidean measure $d\mu$ on $\sqrt{-1}.\mathfrak{t}^*$ be so normalized that the volume of a fundamental bloc for the root lattice is equal to one. If the polynomial π_L on $\sqrt{-1}.\mathfrak{t}^*$ is defined by $\pi_L(\lambda) = \prod_{\alpha\in\Delta_L^+} (\alpha,\lambda)$, then Weyl's integral formula says that the Euclidean measure $d\nu$ on $\sqrt{-1}.\ell^*$ can be so normalized that for all $f \in C_c(\sqrt{-1}.\ell^*)$

$$\int_{\sqrt{-1}.\ell^*} f(\nu)\,d\nu = \int_{C_L^+} \pi_L(\mu)^2 \int_{Ad(L)} f(Ad(l)\mu)\,dl\,d\mu$$

In rank one this formula is nothing but a rewriting of an integral over a Euclidean space by means of polar coordinates. The natural projection of $\sqrt{-1}.k^*$ onto $\sqrt{-1}.\ell^*$ is denoted by p_L. Then for a general point λ in $\sqrt{-1}.\mathfrak{t}^*$ the push-forward under p_L of the invariant measure supported by the orbit $Ad(K)\lambda$ is equal to integration against a locally summable Ad(L)-invariant function $D_\lambda^{K,L} : \sqrt{-1}.\ell^* \longrightarrow \mathbb{R}$. We denote the Weyl dimension polynomials for K and L by d_K and d_L respectively.

Theorem 5. For almost all $\mu \in \sqrt{-1}.\mathfrak{t}^*$ we have

$$d_L(\mu).M_\lambda^{K,L}(\mu) = d_K(\lambda).\pi_L(\mu)^2 D_\lambda^{K,L}(\mu)$$

In Kirillov's terminology, the canonical measure on the orbit $Ad(K).\lambda$ has total mass equal to $d_K(\lambda)$, and similarly for Ad(L)-orbits. In view of Weyl's integral formula the above theorem says that the push-forward under p_L of the canonical measure on $Ad(K).\lambda$ is equal to integration over $\mu \in C_L^+$ of the canonical measure on $Ad(L).\mu$, with $M_\lambda^{K,L}(\mu)$ as weight function. So $M_\lambda^{K,L}$, in stead of $m_\lambda^{K,L}$, is the correct function in order to obtain the functorial property of the orbit method. For a connected nilpotent Lie group the Jacobian of the exponential mapping is identically one, which is in general not

true in the compact case. However, the Jacobian of the exponential map is always one at the identity. This may suggest to look at the germ of the character at the identity, which is equivalent to the study of asymptotic behaviour of multiplicities.

In Chapter 4 several examples are treated in detail. In the special case where L is equal to T, it can be deduced from the relation between the projection of orbits and the asymptotic behaviour of multiplicities that the projection $p_T(Ad(K)\lambda)$ is equal to the convex hull of the Weyl group orbit $W.\lambda$. This result has been obtained by B. Kostant several years ago [19]. In Chapter 1 we present a simplified proof without using representation theory. In Chapter 2 we refine this proof in order to obtain more insight in the structure of the orbit spaces.

CHAPTER 1

THE CONVEXITY THEOREMS OF KOSTANT

1.1 Introduction

Let G be a connected real non-compact semisimple Lie group with Lie algebra $\mathfrak{g}$. Fix a Cartan involution θ of $\mathfrak{g}$, and let $\mathfrak{k} = \{X \in \mathfrak{g}: \theta X = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g}: \theta X = -X\}$. We denote the corresponding Cartan involution of G also by θ. The fixed point group K of θ is a connected closed subgroup of G with Lie algebra $\mathfrak{k}$. The exponential map is a diffeomorphism from $\mathfrak{p}$ onto $\{g \in G: \theta g = g^{-1}\}$, whose inverse is denoted by log, and we have the Cartan decomposition $G = K \exp(\mathfrak{p})$.

Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and write $A = \exp(\mathfrak{a})$. Let M be the centralizer and W the normalizer of $\mathfrak{a}$ in K. The Weyl group $W/_M$ acts on A by conjugation and on $\mathfrak{a}$ by the adjoint representation as a finite reflection group. Consider the orthogonal projection p of $\mathfrak{p}$ onto $\mathfrak{a}$ with respect to the Killing form $B(.,.)$.

Theorem 1. For each $H_0 \in \mathfrak{a}$ the orthogonal projection of the orbit $Ad(K)H_0$ on $\mathfrak{a}$ is equal to the convex hull of the Weyl group orbit $Ad(W)H_0$.

Choose an ordering on the set of roots Δ of the pair $(\mathfrak{g},\mathfrak{a})$. Put $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$ where $\mathfrak{g}^\alpha = \{X \in \mathfrak{g}: [H,X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$ and let $N = \exp(\mathfrak{n})$. According to the Iwasawa decomposition $G = KAN$ we can write each $g \in G$ in the form $g = kan$ with $k \in K$, $a \in A$ and $n \in N$. The Iwasawa projection $H : G \to \mathfrak{a}$ is defined by $H(g) = \log(a)$.

Theorem 2. For each $a \in A$ the Iwasawa projection of $\{kak^{-1}: k \in K\}$ is equal to the convex hull of $Ad(W)\log(a)$.

Both theorems are due to B. Kostant [19]. In this chapter we want to present a fairly easy proof of these theorems. For the proof in section 3 of the infinitesimal case we compute in section 2 the stationary set (set of critical points) and the maxima of a certain function ϕ^{H,H_0}. This function has been introduced before by G.A. Hunt [15] to prove Cartan's conjugacy theorem $\mathfrak{p} = Ad(K)\mathfrak{a}$.

In section 4 we prove the global case using a function ψ^{H,H_0} analogous to ϕ^{H,H_0}. The stationary set of ψ^{H,H_0} is equal to the stationary set of ϕ^{H,H_0}. Because the Hessian of ψ^{H,H_0} is more difficult to handle we reduce the global case by a homotopy argument to the infinitesimal case. The stationary points of ψ^{H,H_0} were first studied by J.J. Duistermaat, J.A.C. Kolk and V.S. Varadarajan [8] to prove an asymptotic formula for the elementary spherical functions as the parameter tends to infinity. In order to apply the method of the stationary phase they computed the stationary set (Lemma 1.5) and the Hessian of the function ψ^{H,H_0}. So the proof of Theorem 1 also holds for Theorem 2.

The results of this chapter have been obtained following a suggestion of J.J. Duistermaat and V.S. Varadarajan.

1.2 The stationary points and the Hessian of the function ϕ^{H,H_0}.

Define for $H,H_0 \in \mathfrak{a}$ the function ϕ^{H,H_0} on K by

$$\phi^{H,H_0}(k) = B(H,Ad(k)H_0) \qquad (k \in K)$$

Using the power series expansion of $Ad(\exp tX) = \exp(ad\, tX)$ one easily sees:

$$\frac{d}{dt}\{\phi^{H,H_0}(k.\exp(tX))\}_{t=0} = B(Ad(k^{-1})H,[X,H_0])$$

$$\frac{d^2}{dt^2}\{\phi^{H,H_0}(k.\exp(tX))\}_{t=0} = B(Ad(k^{-1})H,[X,[X,H_0]])$$

$$-\varphi^{H,H_0}(k) + \tfrac{1}{2}(H,H) + \tfrac{1}{2}(H_0,H_0) = \tfrac{1}{2}(H - Ad(k)H_0,\ H - Ad(k)H_0)$$

for $k \in K$, $X \in k$.

For any Lie subgroup G_1 of G we denote $\{g \in G_1 : Ad(g)H = H\}$ by G_1^H. The Lie algebra of G_1^H is $g_1^H = \{X \in g_1 : [X,H] = 0\}$ where g_1 is the Lie algebra of G_1. We write $(G_1)^o$ for the connected component of G_1 containing the identity.

Lemma 1.1 $K^H = M.(K^H)^o$ for $H \in a$.

Proof: The Lie algebra of $(G^H)^o$ is $g^H = k^H \oplus p^H$ and $(G^H)^o = (K^H)^o.\exp(p^H)$ is a Cartan decomposition for $(G^H)^o$. Take $k \in K^H$. Then $Ad(k^{-1})a$ is a maximal abelian subspace of p^H. Applying Cartan's conjugacy theorem to $(G^H)^o$ there exists $l \in (K^H)^o$ with $Ad(l.k^{-1})a = a$, which implies $l.k^{-1} \in W^H$. Because $Ad(W) \simeq W/_M$ is a finite group generated by reflections, it is well-known that $Ad(W^H) \simeq W^H/_M$ is generated by those reflections in $Ad(W)$, which stabilize H. But this group is the Weyl group $(W \cap (K^H)^o)/_M$ of the space $(G^H)^o/(K^H)^o$, hence $W^H = M.(W \cap (K^H)^o)$. Now the lemma follows because $k \in W^H.(K^H)^o = M.(W \cap (K^H)^o).(K^H)^o = M.(K^H)^o$. □

Lemma 1.2 The set of stationary points of ϕ^{H,H_0} is equal to the set $K^H.W.K^{H_0} = (K^H)^o.W.(K^{H_0})^o$.

Proof: Because $[H_0, Ad(k^{-1})H] \subset [p,p] \subset k$, the condition for $k \in K$ to be a stationary point of ϕ^{H,H_0} is equivalent to $[H_0, Ad(k^{-1})H] = 0$. Suppose $k \in K$ is a stationary point of ϕ^{H,H_0}, i.e. $[H_0, Ad(k^{-1})H] = 0$. By the conjugacy theorem there exists $z \in K^{H_0}$ such that $Ad(z)Ad(k^{-1})H \in a$. Because $Ad(K)H \cap a = Ad(W)H$ we get $Ad(w)Ad(z)Ad(k^{-1})H = H$ for some $w \in W$. Hence $w.z.k^{-1} \in K^H$, i.e. $k \in K^H.W.K^{H_0}$.

Conversely, let $k = y.w.z$ with $y \in K^H$, $w \in W$ and $z \in K^{H_0}$. Then one can check immediatly that $[H_0, Ad(k^{-1})H] = Ad(z^{-1})[H_0, Ad(w^{-1})H] = 0$. □

For a root $\alpha \in \Delta$ we denote $V_\alpha = \{H \in a : \alpha(H) = 0\}$ and s_α the orthogonal reflection in the hyperplane V_α. The complement in a of

the V_α's is a finite union of polyhedral cones, the so-called Weyl chambers. It is well-known that the closure of a Weyl chamber is a fundamental domain for the action of the Weyl group on a.

Lemma 1.3 Suppose $H_1, H_2 \in a$. If $\alpha(H_1)\alpha(H_2) \geqq 0$ for all $\alpha \in \Delta$, then there exists a Weyl chamber C with $H_1, H_2 \in \overline{C}$.

Proof: Suppose $\alpha(H_1)\alpha(H_2) \geqq 0$ for all $\alpha \in \Delta$. Choose a Weyl chamber C_1 with $H_1 \in \overline{C_1}$ and let $\Delta_1^+ = \{\alpha \in \Delta : \alpha(C_1) > 0\}$. Now we construct C by induction on the cardinality of $\{\alpha \in \Delta_1^+ : \alpha(H_2) < 0\}$.

If $\alpha(H_2) \geqq 0$ for all $\alpha \in \Delta_1^+$, we can take $C = C_1$. Otherwise we can choose a simple root $\beta \in \Delta_1^+$ with $\beta(H_2) < 0$. Because $\beta(H_1)\beta(H_2) \geqq 0$ and $\beta(H_1) \geqq 0$, we have $\beta(H_1) = 0$. Therefore $C_2 = s_\beta C_1$ is a Weyl chamber with $H_1 \in \overline{C_2}$, and for the corresponding positive system $\Delta_2^+ = s_\beta \Delta_1^+$ the cardinality of $\{\alpha \in \Delta_2^+ : \alpha(H_2) < 0\}$ is one less than the cardinality of $\{\alpha \in \Delta_1^+ : \alpha(H_2) < 0\}$. This proves the lemma. □

Lemma 1.4 The following statements are equivalent:

a. ϕ^{H,H_0} has a local maximum at $k \in K$.
b. $k = y.w.z$ for some $y \in K^H$, $w \in W$, $z \in K^{H_0}$ and there exists a Weyl chamber C with H, $Ad(w)H_0 \in \overline{C}$.
c. ϕ^{H,H_0} has an absolute maximum at $k \in K$.

Proof:

a ⇒ b: Suppose ϕ^{H,H_0} has a local maximum at $k \in K$. By Lemma 1.2 we have $k = y.w.z$ for some $y \in K^H$, $w \in W$ and $z \in K^{H_0}$. Because ϕ^{H,H_0} is left-invariant under K^H and right-invariant under K^{H_0}, w is also a local maximum for ϕ^{H,H_0}. Therefore

$$\frac{d^2}{dt^2} \{ \phi^{H,H_0}(w.\exp(tX)) \}_{t=0} = B(Ad(w^{-1})H, [X,[X,H_0]]) \leqq 0$$

for all $X \in k$. Choose for every root $\alpha \in \Delta$ elements $X_\alpha \in g^\alpha$ with length normalized by $B(X_\alpha, \theta X_\alpha) = -1$. Substituting for $X = Ad(w^{-1})(X_\alpha + \theta X_\alpha)$ we get $\alpha(H)\alpha(Ad(w)H_0) \geq 0$, which implies by Lemma 1.3 the existence of a Weyl chamber C with $H, Ad(w)H_0 \in \overline{C}$.

$b \Rightarrow c$: Suppose we have two points $k = y.w.z$ and $k' = y'.w'.z'$ in K for some $y,y' \in K^H$, $w,w' \in W$ and $z,z' \in K^H$, such that there exist Weyl chambers C and C' with $H, Ad(w)H_0 \in \overline{C}$ and $H, Ad(w')H_0 \in \overline{C}'$. Choose $v \in W$ such that $C' = Ad(v)C$. Because the closure of a Weyl chamber is a fundamental domain for the action of W on $\mathfrak{a}$, we get from $H \in \overline{C} \cap \overline{C}'$ that $v \in W^H$. Then $Ad(v^{-1}w')H_0$ and $Ad(w)H_0$ both lie in $\overline{C}$, so $Ad(v^{-1}w')H_0$ is equal to $Ad(w)H_0$. Hence $w' \in W^H.w.W^{H_0}$. Now $\phi^{H,H_0}(k') = \phi^{H,H_0}(w') = \phi^{H,H_0}(w) = \phi^{H,H_0}(k)$. In other words, ϕ^{H,H_0} has the same value at all points $k \in K$ satisfying statement b. Because K/K^{H_0} is compact, there exists an absolute maximum $k \in K$ for ϕ^{H,H_0}. Of course, this k satisfies statement b, so we are done.

$c \Rightarrow a$: trivial. □

1.3 The infinitesimal convexity theorem.

Fix $H_0 \in \mathfrak{a}$. For a subset $V \subset W$ we denote by $\mathfrak{a}(H_0,V)$ the convex hull of $\{Ad(v)H_0 : v \in V\}$. In this section we prove the following theorem

Theorem 1. $p(Ad(K)H_0) = \mathfrak{a}(H_0,W)$.

Proof: Clearly, it is sufficient to prove the theorem for $H_0 \in \mathfrak{a}$ regular. Fix a boundary point $p(Ad(k_0)H_0)$ of $p(Ad(K)H_0)$ for some $k_0 \in K$. Then for $k \in K$ the map $k \to p(Ad(k)H_0)$ cannot be a submersion at k_0, so there exists $H \in \mathfrak{a}$, $H \neq 0$, such that $\frac{d}{dt}\{\phi^{H,H_0}(k_0.\exp(tX))\}_{t=0} = 0$ for all $X \in \mathfrak{k}$. From Lemma 1.2 we see that $k_0 = y.w$ for some $y \in (K^H)^\circ$ and $w \in W$.

Because $K - \bigcup_{H \in \{\mathfrak{a}-0\}} K^H.W$ is the complement in K of a finite union of submanifolds of positive codimension, it is a dense open subset of K. On this dense open subset the map $k \to p(Ad(k)H_0)$ is submersive, hence $p(Ad(K)H_0)$ has dense interior.

Now we use induction on the dimension of G. Although $(G^H)^\circ$ is reductive, the adjoint action of the center of $(G^H)^\circ$ on $\mathfrak{g}^H$ is trivial, and therefore the induction works for $(G^H)^\circ$. We can conclude, by induction, that $p(Ad((K^H)^\circ)Ad(w)H_0)$ is ~~equal~~ contained to $\mathfrak{a}(Ad(w)H_0,W^H)$. Moreover the rank of the map $k \to p(Ad(k.w)H_0)$ is, on a dense open subset of $(K^H)^\circ$, equal to the rank of Δ^H, where $\Delta^H = \{\alpha \in \Delta : \alpha(H) = 0\}$ is the root system of the pair $(\mathfrak{g}^H,\mathfrak{a})$.

Because $p(Ad(K)H_0)$ has dense interior, each connected component of $\mathfrak{a}(H_0,W) - \bigcup_{H \in \{\mathfrak{a}-0\}, w \in W} \mathfrak{a}(Ad(w)H_0,W^H)$ is either completely contained in $p(Ad(K)H_0)$ or has a void intersection with $p(Ad(K)H_0)$.

Perturbing the point $k_0 = y.w$ with $y \in (K^H)^\circ$ and $w \in W$, we may assume that the rank of Δ^H is equal to $\text{rank}(\Delta) - 1$ and that the boundary of $p(Ad(K)H_0)$ is of the form $\mathfrak{a}(Ad(w)H_0,W^H)$ in a small neighbourhood of $p(Ad(k_0)H_0)$. Taking for H the outwards directed normal on the boundary of $p(Ad(K)H_0)$ at $p(Ad(k_0)H_0)$ the function ϕ^{H,H_0} has a local maximum at k_0. Using Lemma 1.4 this local maximum is an absolute maximum, hence $\mathfrak{a}(Ad(w)H_0,W^H)$ must lie in the boundary of $\mathfrak{a}(H_0,W)$. This proves the theorem. □

1.4 The global convexity theorem.

For $H,H_0 \in \mathfrak{a}$ we define the function ψ^{H,H_0} on K by

$$\psi^{H,H_0}(k) = B(H,\mathcal{H}(\exp(Ad(k)H_0))) \qquad (k \in K)$$

The stationary points of ψ^{H,H_0} are the same as those of ϕ^{H,H_0}.

Lemma 1.5 The set of stationary points of ψ^{H,H_0} is equal to $K^H.W.K^{H_0}$.

Proof: Suppose $\exp(Ad(k)H_0) = k_1 a_1 n_1 = k_1 s_1$ with $k_1 \in K$, $a_1 \in A$, $n_1 \in N$ and $s_1 = a_1 n_1$. Then $\mathcal{H}(\exp(Ad(\exp(tX).k)H_0) = \mathcal{H}(k_1 s_1 \exp(-tX)) = \mathcal{H}(\exp(-t.Ad(s_1)X).s_1) = \mathcal{H}(a_1) + \mathcal{H}(\exp(-t.Ad(s_1)X))$, because A normalizes N. So $\frac{d}{dt}\{\psi^{H,H_0}(\exp(tX).k)\}_{t=0} = 0$ for all $X \in k \Leftrightarrow$ $B(H,Ad(s_1)X) = 0$ for all $X \in k \Leftrightarrow B(Ad(n_1^{-1})H,X) = 0$ for all $X \in k \Leftrightarrow$ $B(Ad(n_1^{-1})H - H,X) = 0$ for all $X \in k \Leftrightarrow Ad(n_1^{-1})H = H$ because $Ad(n_1^{-1})H - H \in n$.

Since $k.\exp(2H_0).k^{-1} = \theta(n_1^{-1}).(a_1)^2.n_1$, we have $n_1 \in N^H \Leftrightarrow$ $\theta(n_1^{-1}).(a_1)^2.n_1 \in G^H \Leftrightarrow \exp(2Ad(k)H_0) \in G^H \Leftrightarrow [Ad(k)H_0,H] = 0 \Leftrightarrow$ $k \in K^H.W.K^{H_0}$. □

The next theorem is a global analogue of Theorem 1.

Theorem 2. $H(\exp(Ad(K)H_0)) = a(H_0,W)$.

Proof: Define a homotopy $H_t : p \to a$ of projections by

$$H_t(Z) = \begin{cases} \frac{1}{t} H(\exp(tZ)) & \text{for } 0 < t \leq 1 \\ p(Z) & \text{for } t = 0 \end{cases}$$

Clearly H_t is a continuous homotopy between the Iwasawa projection and the orthogonal projection. If we define the function $\phi_t^{H,H_0}(k) = B(H,H_t(Ad(k)H_0))$, then the stationary set of ϕ_t^{H,H_0} is equal to $K^H.W.K^{H_0}$.

In the proof of Theorem 1 we concluded that $p(Ad(K)H_0)$ was a subset of $a(H_0,W)$ and each component of $a(H_0,W) - \bigcup_{H\in\{a-o\},w\in W} a(Ad(w)H_0,W^H)$ was either completely contained in $p(Ad(K)H_0)$ or had a void intersection with $p(Ad(K)H_0)$. This conclusion was made only by using that $K^H.W.K^{H_0}$ was the stationary set of ϕ^{H,H_0}. Hence this conclusion is also true for the projections H_t , $0 \leq t \leq 1$, in stead of p. For t = 0 we have $H_0(Ad(K)H_0) = a(H_0,W)$ by Theorem 1. The set of $t \in [0,1]$ such that some point in a given component belongs to $H_t(Ad(K)H_0)$ is open using continuity. It is closed using compactness and continuity. This implies that $H_t(Ad(K)H_0) = a(H_0,W)$ for all $t \in [0,1]$. In particular, $H_1(Ad(K)H_0) = H(\exp(Ad(K)H_0)) = a(H_0,W)$. □

CHAPTER 2

A GENERALISATION OF THE CONVEXITY THEOREM

2.1 Introduction.

We use the notation of the previous chapter. Fix a positive Weyl chamber a^+ and let $H_0 \in Clos(a^+)$. If G = KAN is the Iwasawa decomposition corresponding to a^+, then we write for $g \in G$:

$$g = k(g)a(g)n(g) \qquad k(g) \in K,\ a(g) \in A,\ n(g) \in N$$

The subgroup $P = K^{H_0}AN$ is a parabolic subgroup of G, and

$$\begin{array}{ccccc} G/P & \xrightarrow{gP \to k(g)K^{H_0}} & K/K^{H_0} & \xrightarrow{kK^{H_0} \to Ad(k)H_0} & Ad(K)H_0 \\ \uparrow{\scriptstyle g \to gP} & & \uparrow{\scriptstyle k \to kK^{H_0}} & & \uparrow{\scriptstyle Id} \\ G & \xrightarrow[g \to k(g)]{} & K & \xrightarrow[k \to Ad(k)H_0]{} & Ad(K)H_0 \end{array}$$

is a commutative diagram with in the upper row diffeomorphisms. When we are speaking of the flag variety, we mean one of these three objects in the upper row with identifications as above. For example, the orthogonal projection p of the flag variety on a is the restriction to $Ad(K)H_0$ of the orthogonal projection $p : p \to a$, while A acts on the flag variety by left multiplication on G/P.

A remarkable fact is that one can define a Riemannian metric on the flag variety, such that the gradient field of the function ϕ^{H,H_0} is equal to the infinitesimal action of $H \in a$ on the flag variety.

This result, explained in section 2, has interesting consequences. In the first place, it provides a new proof of the Bruhat decomposition. Secondly, it shows that the closure of an A-orbit is the correct object to study the projection from the flag variety on $\mathfrak{a}$. The main result of section 5 is

Theorem 3. The projection p is a bijection from the closure of an A-orbit A_x onto the convex hull of $\{Ad(w)H_0 : Ad(w)H_0 \in \overline{A}_x\}$.

A Schubert variety is the closure of a Bruhat cell. In section 6 we prove that for each Schubert variety S in the flag variety the A-orbits A_x with $\overline{A}_x \cap Ad(W)H_0 = S \cap Ad(W)H_0$ form a dense subset of S. Together with Theorem 3 this implies that the projection p(S) of a Schubert variety S is equal to the convex hull of $\{Ad(w)H_0 : Ad(w)H_0 \in S\}$. This result is a generalisation of Theorem 1.

For a subset $V \subset W$ we denote by $\mathfrak{a}(H_0,V)$ the convex hull of $\{Ad(v)H_0 : v \in V\}$. If $V = \{w\}$ or $V = \{w, s_\alpha w\}$ for some $w \in W$ and $\alpha \in \Delta$, then $\mathfrak{a}(H_0,V)$ is called a root polytope. By induction on the dimension of $\mathfrak{a}(H_0,V)$ we say that for $V \subset W$ with $\dim \mathfrak{a}(H_0,V) \geq 2$ $\mathfrak{a}(H_0,V)$ is a root polytope if all proper faces of $\mathfrak{a}(H_0,V)$ are root polytopes. It is easy to see that for each A-orbit A_x on the flag variety the projection $p(\overline{A}_x)$ is always a root polytope. It seems to me that the converse is also true.

Conjecture: If V is a subset of W, such that $\mathfrak{a}(H_0,V)$ is a root polytope, then there exists an A-orbit A_x on the flag variety with $p(\overline{A}_x) = \mathfrak{a}(H_0,V)$.

Finally it should be mentioned that the study of A-orbits is closely related with the theory of toric varieties in algebraic geometry [6 ,16]. In fact, if G is complex and we consider MA-orbits in stead of A-orbits, then we get in this way many examples of toric varieties.

2.2 A Riemannian metric on the flag variety.

Fix $H_0 \in Clos(\mathfrak{a}^+)$. Let $p_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$ be the projection onto $\mathfrak{k}$ along $\mathfrak{a} + \mathfrak{n}$. Choose for each $\alpha \in \Delta^+$ vectors $X_\alpha \in \mathfrak{g}^\alpha$ with length normalized by $B(X_\alpha, \theta X_\alpha) = -1$. If we put $E_\alpha = \sqrt{\tfrac{1}{2}}.(X_\alpha + \theta X_\alpha)$ and $F_\alpha = -\sqrt{\tfrac{1}{2}}.(X_\alpha - \theta X_\alpha)$, then $E_\alpha \in \mathfrak{k}$, $F_\alpha \in \mathfrak{p}$ and $-B(E_\alpha,E_\alpha) = B(F_\alpha,F_\alpha) = 1$. Moreover $[E_\alpha,H] = \alpha(H).F_\alpha$ for all $H \in \mathfrak{a}$ and $p_{\mathfrak{k}}(F_\alpha) = E_\alpha$.

Consider the linear map $X \to p_{\mathfrak{k}}[X,H_0]$ from $\mathfrak{k}$ into itself. From what we said above it is clear that E_α is an eigenvector with eigenvalue $\alpha(H_0)$. Hence this map is a positive semi-definite symmetric linear map with kernel $\mathfrak{k}^{H_0}$. In other words, we can define a positive semi-definite symmetric bilinear form R^{H_0} on $\mathfrak{k}$ by

$$R^{H_0}(X,Y) = -B(X, p_{\mathfrak{k}}[Y,H_0]) \qquad (X,Y \in \mathfrak{k})$$

Dit is Ad(K^{H_0})-invariant,

Because the radical of R^{H_0} is equal to $\mathfrak{k}^{H_0}$, we can consider R^{H_0} also as an inner product on $\mathfrak{k}/\mathfrak{k}^{H_0}$. By translation we get a K-invariant Riemannian metric on the flag variety K/K^{H_0}. We denote this Riemannian metric also by R^{H_0}.

For $H \in \mathfrak{a}$ we denote the velocity field of the action of the 1-parameter group $t \to \exp(tH)$ on the flag variety G/P by v^{H,H_0}. In section 1.2 we introduced a function ϕ^{H,H_0} on K. Because ϕ^{H,H_0} is right-K^{H_0}-invariant, we can consider ϕ^{H,H_0} also as a function on the flag variety K/K^{H_0}.

Lemma 2.1 The gradient field of ϕ^{H,H_0} with respect to the Riemannian metric R^{H_0} is equal to v^{H,H_0}.

Proof: In view of our identifications there is a one-to-one correspondence between a right-K^{H_0}-invariant function $f \in C^\infty(K)$ and a right-P-invariant function $\tilde{f} \in C^\infty(G)$.

For $k \in K$ we have

$$(v^{H,H_0}f)(k) = \frac{d}{dt}\{ \tilde{f}(\exp(tH).k) \}_{t=0}$$
$$= \frac{d}{dt}\{ \tilde{f}(k.\exp(t.p_{k}(Ad(k^{-1})H))) \}_{t=0}$$
$$= \frac{d}{dt}\{ f(k.\exp(t.p_{k}(Ad(k^{-1})H))) \}_{t=0}$$

Let v be a smooth map from K into k, such that $Ad(k_0)v(k.k_0) = v(k)$ for all $k \in K$, $k_0 \in K^{H_0}$. We consider v as a smooth vector field on $K/_{K}H_0$ by

$$(vf)(k) = \frac{d}{dt}\{ f(k.\exp(t.v(k))) \}_{t=0}$$

where $f \in C^\infty(K)$ is a right-K^{H_0}-invariant function. Clearly $v(f) \in C^\infty(K)$ is again right-K^{H_0}-invariant. Conversely, each smooth vector field on $K/_{K}H_0$ is of this form.

Now for $k \in K$ we have

$$(v\phi^{H,H_0})(k) = \frac{d}{dt}\{ \phi^{H,H_0}(k.\exp(t.v(k))) \}_{t=0}$$
$$= B(Ad(k^{-1})H,[v(k),H_0])$$
$$= -B(p_{k}(Ad(k^{-1})H),p_{k}[v(k),H_0])$$
$$= R^{H_0}(p_{k}(Ad(k^{-1})H),v(k))$$
$$= R^{H_0}(v^{H,H_0}(k),v(k))$$

Here we have used the equality $B(Z_1,Z_2) = -B(p_{k}Z_1,p_{k}Z_2)$ for $Z_1 \in p$ and $Z_2 \in p \cap a^{\perp}$. □

2.3 The Bruhat decomposition.

Let M be a compact Riemannian manifold and $\phi : M \to \mathbb{R}$ a smooth function with finitely many stationary points, say $x_1,x_2,\dots,x_n$. Suppose that all stationary points are non-degenerate, i.e. the Hessian of ϕ at $x_1,\dots,x_n$ is non-degenerate. Let v be the gradient field of ϕ, and $D_t : M \to M$ the corresponding 1-parameter group of diffeomorphisms ($t \in \mathbb{R}$). The set $S_i = \{x \in M: \lim_{t\to\infty} D_t(x) = x_i\}$ is called the stable manifold of v through x_i. The fundamental theorem of the Morse theory says that each S_i is a Euclidean cell of dimension equal to the index of ϕ at x_i and $M = \cup S_i$ is a disjoint union. For more details about Morse theory we refer to Milnor's book [22].

We want to apply this theorem for the flag variety $G/_P$ with the Riemannian metric as defined in section 2.2. For the function on $G/_P$ we take ϕ^{H,H_0}. By Lemma 2.1 the gradient field of ϕ^{H,H_0} is equal to v^{H,H_0}. In order to find the stable manifold of v^{H,H_0} through $w \in W$, we choose special coordinates in a neighbourhood of w. For $w \in W/_{W}H_0$ we consider the nilpotent Lie algebra $b_w = \sum_{\alpha(wH_0)<0} g^\alpha$. The map $\psi_w : b_w \to G/_P$ is defined by

$$\psi_w(X) = \exp(X).w.P \qquad (X \in b_w)$$

The image of ψ_w is denoted by B_w.

<u>Lemma 2.2</u> B_w is an open subset of $G/_P$ and ψ_w is a diffeomorphism of b_w onto B_w.

Proof: Par transport de structure it suffices to prove the lemma for w is the identity e. Suppose $\psi_e(X_1) = \psi_e(X_2)$ for $X_1,X_2 \in b_e$. Writing $\exp(X) = \exp(-X_1).\exp(X_2)$ for some $X \in b_e$, this implies that $\exp(X) \in P$. Because P is normalized by A, we have $\exp(e^{t.ad(H)}X) \in P$ for all $H \in a$, $t \in \mathbb{R}$.

Fix $H \in a^+$. Then $\lim_{t\to\infty} e^{t.ad(H)}X = 0$, and so $e^{t.ad(H)}X$ lies in the Lie algebra of P for t sufficiently large, i.e. $e^{t.ad(H)}X$ lies in $k^{H_0} + a + n$ for large t. Because of the decomposition $g = b_e + k^{H_0} + a + n$, we see that $X = 0$, which proves that ψ_e is injective. It follows easily that $(d\psi_e)_X$ is injective for all $X \in b_e$. Since the dimensions of $\mathfrak{b}_e$ and $G/_P$ are the same, $(d\psi_e)_X$ is bijective for all $X \in \mathfrak{b}_e$. Now the lemma follows from the inverse function theorem. □

The next lemma is clear from the definition of the vector field v^{H,H_0}.

<u>Lemma 2.3</u> The pull-back under ψ_w of the vector field $v^{H,H_0}|_{B_w}$ becomes the linear vector field $ad(H) : \mathfrak{b}_w \to \mathfrak{b}_w$. In particular, B_w is complete for the flow on $G/_P$ corresponding to the vector field v^{H,H_0}.

<u>Corollary 1</u> Suppose $-H \in \mathfrak{a}^+$. The stable manifold of v^{H,H_0} through $w \in W/_{W^{H_0}}$ is the N-orbit through w.

Proof: Clearly the stable manifold of v^{H,H_0} through w is a subset of B_w. The condition for $X \in \mathfrak{b}_w$ that $\psi_w(X)$ lies in the stable manifold of v^{H,H_0} through w is transferred by ψ_w into $\lim_{t\to\infty} e^{t.ad(H)}X = 0$. But this is equivalent to $X \in \mathfrak{b}_w \cap \mathfrak{n}$. Hence the corollary follows because the N-orbit through w is equal to $\psi_w(\mathfrak{b}_w \cap \mathfrak{n})$. □

Waarom?

<u>Corollary 2</u> (Bruhat decomposition)

$$G = \bigcup_{w \in W/_{W^{H_0}}} N.w.P \quad \text{is a disjoint union.}$$

<u>Remark:</u> For H_0 regular this decomposition has been obtained by F. Bruhat for the complex classical groups [4]. A general proof was given by Harish-Chandra [10]. The proof we gave above was suggested to me by J.J. Duistermaat and will also appear in a somewhat different context in [8]. The basic idea that Bruhat cells are the stable manifolds of a gradient vector field is due to R. Hermann [13].

<u>Corollary 3</u> The set $\{(\mathfrak{b}_w, \psi_w, B_w) : w \in W/_{W^{H_0}}\}$ is a coordinate covering of $G/_P$.

2.4 Closed convex cones.

Let E be a Euclidean space with inner product (.,.). A subset K of E is called a cone (with top 0) if for each $X \in K$ we have $t.X \in K$ for all $t \in \mathbb{R}^+$. From now on K is a closed convex cone. The dimension of K is by definition the dimension of the linear space $\mathbb{R}.K$ spanned by K. It is easy to see that K has non-empty interior relative to $\mathbb{R}.K$. The relative interior of K is denoted by Relint(K).

Let $K^\vee = \{X \in E: (X,Y) \geq 0 \text{ for all } Y \in K\}$. Then $K^\vee$ is again a closed convex cone, the so-called dual cone of K. A subset F of K is called a face of K if $F = X^\perp \cap K$ for some $X \in K^\vee$. Clearly, if K' is a closed convex cone contained in K and F is a face of K, then $F \cap K'$ is a face of K'. Faces are closed convex cones and the intersection of two faces is again a face. If a face F of K contains a point of Relint(K), then F = K. All other faces of K are called proper faces of K. The dimension of a proper face of K is less than the dimension of K.

<u>Lemma 2.4</u> Suppose $X \in K$. Then there exists a unique face F of K with $X \in$ Relint(F).

Proof: Suppose F_1 is a face of K with $X \in F_1$. If all faces F of K, which contain X, also contain F_1, then the lemma is proved. Otherwise there exists a face F_2' of K with $X \in F_2'$ but $F_1 \not\subset F_2'$. Then $F_2 = F_1 \cap F_2'$ is a face of K with $X \in F_2$, and $F_2 \subsetneqq F_1$. Repeating this procedure at the most finitely many times we end up with a face F_k of K with $X \in F_k$ and all faces F of K, which contain X, also contain F_k. □

<u>Lemma 2.5</u> Suppose $\{X_n, n \in \mathbb{N}\}$ is a sequence in E such that $\lim_{n\to\infty} e^{(X,X_n)}$ exists for all $X \in K$. Then there exist $Y \in E$, $Z \in K^\vee$ such that

$$\lim_{n\to\infty} e^{(X,X_n)} = \lim_{t\to\infty} e^{(X,Y-t.Z)}$$

for all $X \in K$.

Proof: Suppose $K' = \{X \in K : \lim_{n\to\infty} e^{(X,X_n)} \neq 0\}$. Then $K' = \mathbb{R}.K' \cap K$, hence K' is a closed convex cone. Choose $X \in \mathrm{Relint}(K')$. Then there exists a unique face F of K with $X \in \mathrm{Relint}(F)$. Clearly $F \cap K'$ is a face of K'. Because $(F \cap K') \cap \mathrm{Relint}(K') \neq \emptyset$, we have $K' = F \cap K'$. Hence $K' \subset F$.

Conversely, let $B_\varepsilon(X)$ be a ball in $\mathbb{R}.F$ with radius $\varepsilon > 0$ and center X, such that $B_\varepsilon(X) \subset \mathrm{Relint}(F)$. For $X' \in F \cap B_\varepsilon(0)$ we have

$$\lim_{n\to\infty} e^{(X,X_n)} = \lim_{n\to\infty} e^{(X',X_n)} . \lim_{n\to\infty} e^{(X-X',X_n)}$$

Clearly all three limits exist. Because the left-hand side is non-zero, we get $\lim_{n\to\infty} e^{(X',X_n)} \neq 0$ for all $X' \in F \cap B_\varepsilon(0)$. Hence $B_\varepsilon(0) \cap F \subset K'$, and therefore $F \subset K'$. So we have proved that $K' = F$ is a face of K. Choose $Z \in K^\vee$, such that $K' = Z^\perp \cap K$.

The map $X \to \lim_{n\to\infty} e^{(X,X_n)}$ from K' to $\mathbb{R}^+$ can be extended to a group homomorphism of $\mathbb{R}.K'$ into $\mathbb{R}^+$. So there exists $Y \in \mathbb{R}.K'$ with $\lim_{n\to\infty} e^{(X,X_n)} = e^{(Y,X)}$ for all $X \in \mathbb{R}.K'$.

For $X \in K'$ we have

$$\lim_{t\to\infty} e^{(X,Y-t.Z)} = e^{(X,Y)} = \lim_{n\to\infty} e^{(X,X_n)}$$

and for $X \in K \setminus K'$

$$\lim_{t\to\infty} e^{(X,Y-t.Z)} = 0 = \lim_{n\to\infty} e^{(X,X_n)} \quad . \quad \square$$

2.5 The projection of the closure of an A-orbit.

Fix $x \in G/_P$ and consider the integral curve of v^{H,H_0} through x for some $H \in \mathfrak{a}$. We can choose $w_1 \in W$ such that $-H \in \mathrm{Clos}(\mathrm{Ad}(w_1)\mathfrak{a}^+)$. Using the Bruhat decomposition $G = \bigcup_{w \in W/_W H_0} w_1.N.w_1^{-1}.w.P$, we can write x in

the form $x = \exp(\mathrm{Ad}(w_1)X).w.P$ for some $X \in \mathfrak{n}$, $w \in W$. Clearly the point $y = \exp\{\mathrm{Ad}(w_1)(\lim_{t\to\infty} e^{t.\mathrm{ad}(\mathrm{Ad}(w_1^{-1})H} X)\}.w.P$ is a well-defined point in $G/_P$. Moreover, y is the unique end point of the integral curve of v^{H,H_0} through x. Similarly, the integral curve of v^{H,H_0} through x has also a unique begin point (Just take the endpoint of the integral curve of v^{-H,H_0} through x).

Consider the A-orbit A_x through x. The action of A on $\overline{A}_x$ commutes with the vector field v^{H,H_0}, because A is abelian. So we have the following lemma.

Lemma 2.6 The end points of the integral curves of v^{H,H_0} in A_x form a single A-orbit $A_y \subset \overline{A}_x$.

Lemma 2.7 As H ranges over $\mathfrak{a}$ we get in the way of Lemma 2.6 all of $\overline{A}_x$ as a finite union of A-orbits.

Proof: Suppose $z \in \overline{A}_x$. Choose coordinates (b_w, ψ_w, B_w) with $z \in B_w$. Since B_w is open and A-invariant, we have $A_x \subset B_w$. If we write $X = \psi_w^{-1}(x)$ and $Z = \psi_w^{-1}(z)$, then Z lies in the closure of the orbit $\{e^{\mathrm{ad}(H)}X : H \in \mathfrak{a}\}$. So there exists a sequence $\{H_n, n \in \mathbb{N}\}$ in $\mathfrak{a}$ with $\lim_{n\to\infty} e^{\mathrm{ad}(H_n)}X = Z$.

For $Y \in b_w$ we write $\Delta(Y)$ for the set $\{\alpha \in \Delta : Y_\alpha \neq 0\}$ where $Y = \sum_{\alpha(wH_0)<0} Y_\alpha$ with $Y_\alpha \in \mathfrak{g}^\alpha$. Furthermore the convex polyhedral cone $\{ \sum_{\alpha \in \Delta(Y)} r_\alpha.\alpha : r_\alpha \in \mathbb{R}^+ \}$ in $\mathfrak{a}^*$ is denoted by $\mathbb{R}^+.\Delta(Y)$.

Clearly $\lim_{n\to\infty} e^{\mathrm{ad}(H_n)}X = Z$ implies that $\lim_{n\to\infty} e^{\alpha(H_n)} = \|Z_\alpha\| : \|X_\alpha\|$ for all $\alpha \in \Delta(X)$. By Lemma 2.5 we can choose $H, H' \in \mathfrak{a}$ such that $\lim_{n\to\infty} e^{\alpha(H_n)} = \lim_{n\to\infty} e^{\alpha(H'+t.H)}$ for all $\alpha \in \Delta(X)$. This proves the lemma. $\square$

Lemma 2.8 The function ϕ^{H,H_0} has on $\overline{A}_x$ just one local maximal value. In fact, z is a local maximum for $\phi^{H,H_0}|_{\overline{A}_x}$ if z lies in the closure of the A-orbit A_y from Lemma 2.6.

Proof: Suppose $z \in \overline{A}_x$ is a local maximum for ϕ^{H,H_0} on $\overline{A}_x$. Then $v^{H,H_0}(z) = 0$ because ϕ^{H,H_0} is monotonically increasing on each non-constant integral curve of v^{H,H_0}. Choose a sequence $\{x_n, n \in \mathbb{N}\}$ in A_x with $\lim_{n\to\infty} x_n = z$. By Lemma 2.6 the end points y_n of the integral curves of v^{H,H_0} through x_n lie in A_y. If we choose coordinates (b_w, ψ_w, B_w) with $z \in B_w$, then it follows from the next lemma that $\lim_{n\to\infty} y_n = z$. Hence z lies in the closure of A_y. Moreover ϕ^{H,H_0} is constant on A_y because $v^{H,H_0} = 0$ on A_y. So ϕ^{H,H_0} has the same value at all local maxima of $\phi^{H,H_0}|_{\overline{A}_x}$. This proves the lemma. □

Lemma 2.9 Let V be a Euclidean space, $A : V \to V$ a symmetric linear map and $W \subset V$ a closed subset invariant under the flow $x \to e^{t.A}(x)$, $t \in \mathbb{R}$. Suppose $z \in W$, such that there does not exist a non-constant integral curve in W with begin point z. Then, for each sequence $\{x_n, n \in \mathbb{N}\}$ in W with $\lim_{n\to\infty} x_n = z$, we have

$$\lim_{n\to\infty} \{ \lim_{t\to\infty} e^{t.A}(x_n) \} = z$$

Proof: Suppose the lemma is false. Then there exists $\varepsilon > 0$ and a sequence $\{x_n, n \in \mathbb{N}\}$ in W with $\lim_{n\to\infty} x_n = z$, but $\limsup_{t\to\infty} d(e^{t.A}(x_n), z) \geqq \varepsilon$ for all n. Choose $t_n \in \mathbb{R}^+$ with $d(e^{t_n.A}(x_n), z) = \frac{\varepsilon}{2}$. Clearly $\lim_{n\to\infty} t_n = \infty$.

We have the decomposition $V = V^+ + V^0 + V^-$ with $A|_{V^+}$ positive definite, $A|_{V^-}$ negative definite and $V^0 = \mathrm{Ker}(A)$. For $x \in V$ we write $x = x^+ + x^0 + x^-$ with $x^+ \in V^+$, $x^0 \in V^0$, and $x^- \in V^-$.

The sequence $y_n = e^{t_n.A}(x_n) = e^{t_n.A}(x_n^+) + x_n^0 + e^{t_n.A}(x_n^-)$ is a bounded sequence in W, hence after choosing a suitable subsequence $y = \lim_{n\to\infty} y_n$ exists. Clearly $y = y^+ + y^0$ with $y^0 = \lim_{n\to\infty} x_n = z^0 = z$ and $y^+ \neq 0$. So the integral curve $t \to e^{t.A}(y)$ is non-constant, lies in W and has begin point z, which contradicts the assumptions. □

Lemma 2.10 The projection of the tangent space $T_x(A_x)$ to A_x at x on $\mathfrak{a}$ is equal to $\{H \in \mathfrak{a}: v^{H,H_0}(x) = 0\}^{\perp}$.

Proof: $\{\text{the projection of } T_x(A_x)\}^{\perp} =$

$\{H \in \mathfrak{a}: (v^{H',H_0}\phi^{H,H_0})(x) = 0 \text{ for all } H' \in \mathfrak{a}\} =$

$\{H \in \mathfrak{a}: R^{H_0}(v^{H',H_0}, v^{H,H_0})(x) = 0 \text{ for all } H' \in \mathfrak{a}\} =$

$\{H \in \mathfrak{a}: v^{H,H}(x) = 0\}$. □

Theorem 3. The projection p is a bijection from $\overline{A}_x$ onto the convex hull of $\{\mathrm{Ad}(w)H_0 : \mathrm{Ad}(w)H_0 \in \overline{A}_x\}$.

Proof: Use induction on the dimension of A_x.

If $\dim(A_x) = 0$, then $v^{H,H_0}(x) = 0$ for all $H \in \mathfrak{a}$. By Lemma 2.1 and Lemma 1.2 we have $A_x = \mathrm{Ad}(w)H_0$ for some $w \in W$. So, in this case, there is nothing to prove.

Now, suppose that $\dim(A_x) \geqq 1$. Because A is abelian it follows from Lemma 2.10 that $p(A_x)$ is contained in $p(T_x(A_x))$. Moreover p is a submersion on A_x, so $p(A_x)$ consists of interior points of $p(T_x(A_x))$. The set $\overline{A}_x \setminus A_x$ is a finite union of A-orbits A_y of dimension strictly lower than $\dim(A_x)$. By induction $p(\overline{A}_y)$ is a convex polytope for each A-orbit $A_y \subset \overline{A}_x \setminus A_x$. Hence $p(\overline{A}_x)$ is a compact subset of $p(T_x(A_x))$ with dense interior and bounded by a finite number of convex polytopes, so $p(\overline{A}_x)$ is also a polytope. If a codimension 1 hyperplane bounds $p(\overline{A}_x)$ somewhere locally, then by Lemma 2.8 all of $p(\overline{A}_x)$ lies on one side of that hyperplane. So $p(\overline{A}_x)$ is convex as an intersection of half spaces.

To prove that p is injective, one has to remark that p maps A_X onto the interior of $p(\overline{A}_X)$ and $\overline{A}_X \setminus A_X$ onto the boundary of $p(\overline{A}_X)$. By Lemma 2.7 and Lemma 2.8 and induction, p is injective on $\overline{A}_X \setminus A_X$. On the other hand, suppose $y,z \in A_X$ with $p(y) = p(z)$. Choose $H \in \mathfrak{a}$ such that there exists an integral curve of V^{H,H_0} through y and z. Clearly $p(y) = p(z)$ implies that $\phi^{H,H_0}(y) = \phi^{H,H_0}(z)$. So $y = z$ because ϕ^{H,H_0} is monotonically increasing along non-constant integral curves of V^{H,H_0}. □

2.6 Schubert varieties.

Again we consider the flag variety $G/P = Ad(K)H$ for some fixed $H_0 \in Clos(\mathfrak{a}^+)$. A Bruhat cell is a $w.N.w^{-1}$-orbit on G/P for some $w \in W$, and the closure of a Bruhat cell is called a Schubert variety. If $G = Sl(n,\mathbb{C})$ and $P \subset G$ a maximal parabolic subgroup, this generalizes the classical notion of Schubert variety for the Grassmannian G/P.

We write

$$Ad(w_1)H_0 \xrightarrow{\alpha} Ad(w_2)H_0 \text{ for } w_1, w_2 \in W \text{ and } \alpha \in \Delta^+$$

$$\Leftrightarrow$$

$$Ad(w_2)H_0 = Ad(s_\alpha w_1)H_0 \text{ and } \alpha(Ad(w_1)H_0) > 0$$

$$\Leftrightarrow$$

$$Ad(w_2)H_0 = Ad(s_\alpha w_1)H_0 \text{ and } B(H,Ad(w_1)H_0) > B(H,Ad(w_2)H_0) \text{ for all } H \in \mathfrak{a}^+$$

Define a partial order $\leqq$ on $Ad(W)H_0$ by

$$Ad(w_1)H_0 \leqq Ad(w_2)H_0 \quad \Leftrightarrow \quad \begin{cases} \text{there exist } \alpha_1,\ldots,\alpha_k \in \Delta^+ \text{ such that} \\ Ad(w_1)H_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_k} Ad(w_2)H_0 \end{cases}$$

This ordering is called the Bruhat ordering because of the following lemma, a proof of which can also be found in [3].

Lemma 2.11 $Ad(w_1)H_0 \leqq Ad(w_2)H_0$ if and only if $w_1.P \in Clos(N.w_2.P)$, where $Clos(N.w_2.P)$ denotes the closure of the N-orbit through $w_2.P$ in G/P.

Proof: Suppose $Ad(w_1)H_0 \leqq Ad(w_2)H_0$. It suffices to prove the lemma if $Ad(w_1)H_0 \xrightarrow{\alpha} Ad(w_2)H_0$ for $\alpha \in \Delta^+$. Then $w_1.P \in Clos(N.w_2.P)$ follows from a rank 1 consideration, where the flag variety is a sphere. In fact, $\lim_{t\to\infty} exp(t.X).w_2.P = w_1.P$ for all $X \in \mathfrak{g}^\alpha \setminus \{0\}$.

Conversely, if $w_1.P \in Clos(N.w_2.P)$, then $B_{w_1} \cap N.w_2.P \neq \emptyset$. So, there exists $x \in N.w_2.P$ with $Ad(w_1)H_0 \in \overline{A}_X$. The lemma follows if we can find $\alpha \in \Delta^+$ such that $Ad(s_\alpha w_1)H_0 \in \overline{A}_X$ and $Ad(w_1)H_0 \xrightarrow{\alpha} Ad(s_\alpha w_1)H_0$. Suppose, on the contrary, that for no 1-dimensional face $\{t.Ad(w_1)H_0 + (1-t)Ad(s_\alpha w_1)H_0 : t \in [0,1]\}$ of the polytope $p(\overline{A}_X)$ we have $Ad(w_1)H_0 \xrightarrow{\alpha} Ad(s_\alpha w_1)H_0$. Then $\phi^{H,H_0}|_{\overline{A}_X}$ has a local maximum at $Ad(w_1)H_0$ for all $H \in \mathfrak{a}^+$. Applying Lemma 2.8 we get $Ad(w_1)H_0 = Ad(w_2)H_0$. □

Lemma 2.12 The transition functions for the coordinate covering $\{(b_w,\psi_w,B_w): w \in W\}$ of G/P are rational.

Proof: It suffices to prove that $\psi_{s_\alpha}^{-1} \circ \psi_e : b_e \to b_{s_\alpha}$ is a rational map for α a simple root of $\{\beta \in \Delta : \beta(H_0) < 0\}$. Consider the following diagram

$$\begin{array}{ccc} b_e & \xrightarrow{\psi_{s_\alpha}^{-1} \circ \psi_e} & b_{s_\alpha} \\ \uparrow \sigma & & \uparrow \tau \\ \left(\sum_{\substack{\beta\in\Delta,\beta(H_0)<0,\\ \beta\neq\alpha,\beta\neq2\alpha}} \mathfrak{g}^\beta\right)+(\mathfrak{g}^\alpha+\mathfrak{g}^{2\alpha}) & \xrightarrow{id+\rho} & \left(\sum_{\substack{\beta\in\Delta,\beta(H_0)<0,\\ \beta\neq\alpha,\beta\neq2\alpha}} \mathfrak{g}^\beta\right)+(\mathfrak{g}^{-\alpha}+\mathfrak{g}^{-2\alpha}) \end{array}$$

where $\sigma^{-1}(X,Y) = log(exp(X).exp(Y))$ and $\tau(X,Y) = log(exp(X).exp(Y))$. Because b_e and b_{s_α} are nilpotent Lie algebras, σ and τ are polynomial diffeomorphisms [12].

Furthermore, ρ is the transition function in the rank 1 case. Then it follows by SU(2,1) reduction that ρ is rational [12]. Hence $\psi_{s_\alpha}^{-1} \circ \psi_e$ is also rational as a composition of a rational function with polynomial functions. □

Corollary 1 For each Schubert variety S in $G/_P$ there exist A-orbits $A_x \subset S$ with $\overline{A}_x \cap W.P = S \cap W.P$.

Proof: Suppose $S = \mathrm{Clos}(N.w.P)$ for some $w \in W$. Then $w'.P \in S$ if and only if $B_{w'} \cap N.w.P \neq \emptyset$. Because the function $\psi_{w'}^{-1} \circ \psi_w : b_w \to b_{w'}$ is rational, it is well-defined on a Zariski-open subset of $b_w \cap n$. So we can choose $x \in N.w.P$ such that $\overline{A}_x \cap W.P = S \cap W.P$. □

Corollary 2 The orthogonal projection of a Schubert variety S in $\mathrm{Ad}(K)H_0$ on α is equal to the convex hull of $\{\mathrm{Ad}(w)H_0 : \mathrm{Ad}(w)H_0 \in S\}$.

Proof: The Schubert variety S consists of those A-orbits A_x in $G/_P$ for which $\overline{A}_x \cap W.P \subset S \cap W.P$. Hence the proof follows from Theorem 3 and Corollary 1 . □

CHAPTER 3

ON THE FUNCTORIAL PROPERTY OF THE ORBIT METHOD FOR COMPACT LIE GROUPS

3.1 Introduction.

Let K be a compact connected Lie group and L a connected Lie subgroup of K of the same rank. Choose a maximal torus T of K which is also contained in L. The Lie algebras of K, L and T are denoted by k, ℓ and t respectively. Fix an Ad(K)-invariant inner product (.,.) on k. This inner product induces a linear isomorphism between k and k^*, which intertwines the adjoint action of K on k and the coadjoint action of K on k^*. We also identify k^* with $\sqrt{-1}.k^*$ by $f \to \sqrt{-1}.f$ for $f \in k^*$. Let $\Delta_K \subset \sqrt{-1}.t^*$ be the root system of the pair (k,t) , Δ_K^+ a fixed positive system, $C_K^+ = \{\lambda \in \sqrt{-1}.t^* : (\lambda,\alpha) \geqq 0$ for all $\alpha \in \Delta_K^+\}$ the corresponding Weyl chamber, and W_K the Weyl group generated by the reflections s_α for $\alpha \in \Delta_K$. The root system $\Delta_L \subset \sqrt{-1}.t^*$ is a so-called root subsystem of Δ_K, i.e. Δ_L is a subset of Δ_K satisfying

1. $\alpha \in \Delta_L \Rightarrow -\alpha \in \Delta_L$
2. $\alpha,\beta \in \Delta_L$, $\alpha+\beta \in \Delta_K \Rightarrow \alpha+\beta \in \Delta_L$

We put $\Delta_L^+ = \Delta_L \cap \Delta_K^+$, $C_L^+ = \{\lambda \in \sqrt{-1}.t^* : (\lambda,\alpha) \geqq 0$ for all $\alpha \in \Delta_L^+\}$ and W_L the subgroup of W_K generated by the s_α for $\alpha \in \Delta_L$.

Let $\Lambda_u = \{H \in t : \exp(H) = 1\}$ be the unit lattice and $\Lambda_w = \{\lambda \in \sqrt{-1}.t^* : \lambda(H) \in 2\pi\sqrt{-1}.\mathbb{Z}$ for all $H \in \Lambda_u\}$ the weight lattice of T. The root lattice Λ_r is the sublattice of Λ_w generated by Δ_K. For a dominant integral weight $\lambda \in C_K^+ \cap \Lambda_w$ we denote by $\pi(\lambda,K)$ the irreducible representation of K with highest weight λ. The multiplicity function $m_\lambda^{K,L} : C_L^+ \to \mathbb{Z}$ is defined by

$$\pi(\lambda,K)|_L = \sum_{\mu \in C_L^+} m_\lambda^{K,L}(\mu).\pi(\mu,L)$$

If we write (infinitesimally) $K = K_1 \dots K_n$ as a direct product of simple Lie groups, then $L = L_1 \dots L_n$ where $L_i = L \cap K_i$. Moreover $\Delta_K = \Delta_{K_1} \cup \dots \cup \Delta_{K_n}$ is a disjoint union of irreducible root systems and $\Delta_L = \Delta_{L_1} \cup \dots \cup \Delta_{L_n}$ where $\Delta_{L_i} = \Delta_L \cap \Delta_{K_i}$. We denote the weight lattice of Δ_{K_i} by $\Lambda_{w,i}$. Clearly, if we write $\lambda \in \Lambda_w$ in the form $\lambda = \lambda_1 + \dots + \lambda_n$ with $\lambda_i \in \Lambda_{w,i}$, then we have

$$m_\lambda^{K,L}(\mu) = \prod_{i=1}^{n} m_{\lambda_i}^{K_i,L_i}(\mu_i)$$

So, in order to understand the behaviour of the multiplicity function we may assume that K is simple and $L \neq K$.

In section 4 we introduce for $\lambda \in C_K^+$ a piece-wise polynomial function $M_\lambda^{K,L} : \sqrt{-1}.\mathfrak{t}^* \to \mathbb{R}$, which satisfies the relation $M_{t.\lambda}^{K,L}(t.\mu) = t^r.M_\lambda^{K,L}(\mu)$ for $t > 0$ and $r = |\Delta_K^+ \setminus \Delta_L^+| - \text{rank}(\Delta_K^+ \setminus \Delta_L^+)$. The function $M_\lambda^{K,L}$ is called the asymptotic multiplicity function because of the following theorem.

<u>Theorem 4.</u> There exists a constant $C \in \mathbb{R}^+$ such that for $\lambda \in C_K^+ \cap \Lambda_w$ and $\mu \in C_L^+ \cap (\lambda + \Lambda_r)$ we have

$$| m_\lambda^{K,L}(\mu) - M_\lambda^{K,L}(\mu) | \leqq C.(1 + |\lambda|)^{r-1} .$$

We normalize the Euclidean measure $d\mu$ on $\sqrt{-1}.\mathfrak{t}^*$ such that the volume of a fundamental bloc for the root lattice is equal to 1. The polynomial $\pi_L : \sqrt{-1}.\mathfrak{t}^* \to \mathbb{R}$ is defined by $\pi_L(\lambda) = \prod_{\alpha \in \Delta_L^+} (\alpha,\lambda)$. Then the Euclidean measure $d\nu$ on $\sqrt{-1}.\ell^*$ can be so normalized that

$$\int_{\sqrt{-1}.\ell^*} f(\nu)\, d\nu = \int_{C_L^+} \pi_L(\mu)^2 \int_{Ad(L)} f(Ad(l)\mu)\, dl\, d\mu$$

for all $f \in C_c(\sqrt{-1}.\ell^*)$. The orthogonal projection from $\sqrt{-1}.k^*$ onto $\sqrt{-1}.\ell^*$ is denoted by p_L. In section 5 we prove for W_K-regular $\lambda \in \sqrt{-1}.\mathfrak{t}^*$ the existence of a function $D_\lambda^{K,L} : \sqrt{-1}.\ell^* \to \mathbb{R}$, such that

$$\int_{Ad(K)} f(p_L(Ad(k)\lambda))\, dk = \int_{\sqrt{-1}.\ell^*} f(\nu)\, D_\lambda^{K,L}(\nu)\, d\nu$$

for all $f \in C(\sqrt{-1}.\ell^*)$. Moreover, $D_\lambda^{K,L}$ is an Ad(L)-invariant locally summable function on $\sqrt{-1}.\ell^*$ and the support of $D_\lambda^{K,L}$ is equal to $p_L(Ad(K)\lambda)$. We denote the Weyl dimension polynomials for Δ_K^+ and Δ_L^+ by d_K and d_L respectively.

Now $M_\lambda^{K,L}$, in stead of $m_\lambda^{K,L}$, is the correct function to handle, in order to obtain the functorial property of the orbit method.

<u>Theorem 5.</u> For $\lambda \in \text{Int}(C_K^+)$ we have for almost all $\mu \in \sqrt{-1}.\mathfrak{t}^*$

$$d_L(\mu).M_\lambda^{K,L}(\mu) = d_K(\mu).\pi_L(\mu)^2.D_\lambda^{K,L}(\mu)$$

<u>3.2 Partition functions.</u>

Let Λ be a lattice in a Euclidean space E with inner product (.,.). Suppose S is a finite subset of Λ, contained in a half-space, i.e. there exists $\lambda \in E$ such that $(\lambda,\alpha) > 0$ for all $\alpha \in S$. The set of all $\mathbb{Z}$-valued functions on Λ is denoted by V, and $V_S \subset V$ consists of those $f \in V$ for which the support of f is contained in $\bigcup_{i=1}^{n} \{ \lambda_i + \sum_{\alpha \in S} \mathbb{Z}^- . \alpha \}$ for some $\lambda_1, \dots, \lambda_n \in \Lambda$.

We define for $\alpha \in S$ and $\lambda \in \Lambda$ the operators

$$D_\alpha : V \to V \quad \text{by} \quad D_\alpha f(\mu) = f(\mu + \alpha) - f(\mu)$$

$$I_\alpha : V_S \to V_S \quad \text{by} \quad I_\alpha f(\mu) = \sum_{k=0}^{\infty} f(\mu + k\alpha)$$

$$T_\lambda : V \to V \quad \text{by} \quad T_\lambda f(\mu) = f(\mu + \lambda)$$

$$^\vee : V \to V \quad \text{by} \quad f^\vee(\mu) = f(-\mu)$$

The following relations are immediate

1. $D_\alpha D_\beta = D_\beta D_\alpha$ for $\alpha,\beta \in S$
2. $I_\alpha I_\beta = I_\beta I_\alpha$ for $\alpha,\beta \in S$
3. $I_\alpha D_\alpha = D_\alpha I_\alpha = -\mathrm{Id}$ for $\alpha \in S$
4. $T_\lambda D_\alpha = D_\alpha T_\lambda$ for $\alpha \in S$, $\lambda \in \Lambda$
5. $T_\lambda I_\alpha = I_\alpha T_\lambda$ for $\alpha \in S$, $\lambda \in \Lambda$
6. $T_\lambda f^\vee = (T_{-\lambda} f)^\vee$ for $\lambda \in \Lambda$, $f \in V$
7. $T_\lambda T_\mu = T_{\lambda+\mu}$ for $\lambda,\mu \in \Lambda$

The number of ways to write $\mu \in \Lambda$ as a non-negative integral linear combination of elements of S is denoted by $p_S(\mu)$. The function p_S is called the partion function of the set S. For $\lambda \in \Lambda$ we define $\varepsilon_\lambda \in V$ by $\varepsilon_\lambda(\mu) = 0$ if $\mu \neq \lambda$ and $\varepsilon_\lambda(\lambda) = 1$. If S is empty we put $p_S = \varepsilon_0$.

Lemma 3.1 $(p_S)^\vee = (\prod_{\alpha\in S} I_\alpha)\,\varepsilon_0$

Proof: Use induction on the cardinality $|S|$ of S. Choose $\beta \in S$. Then $p_S^\vee(\mu) = p_S(-\mu) = \sum_{k=0}^{\infty} p_{S-\beta}(-\mu-k\beta) = (I_\beta p_{S-\beta}^\vee)(\mu)$. □

Lemma 3.2 $(-1)^{|T|}(\prod_{\alpha\in T} D_\alpha)p_S^\vee = p_{S\setminus T}^\vee$ for $T \subset S$

Proof: This follows immediately from Lemma 3.1 and properties 1, 2 and 3. □

Lemma 3.3 Suppose we have given a function $c \in V$ with finite support. Then the function $f = \sum_{\lambda\in\Lambda} c(\lambda)T_\lambda p_S^\vee$ is the unique solution in V_S of the difference equation

$$(-1)^{|S|}(\prod_{\alpha\in S} D_\alpha)f = \sum_{\lambda\in\Lambda} c(\lambda)\varepsilon_\lambda$$

Proof: It follows immediately from Lemma 3.2 that $f = \sum_{\lambda\in\Lambda} c(\lambda)T_\lambda p_S^\vee$ is a solution. In order to prove that the solution is unique in V_S, we introduce a partial order $\leqq_S$ on Λ by

$$\lambda \leqq_S \mu \iff \mu - \lambda \in \mathbb{Z}^+.S \iff p_S(\mu-\lambda) > 0$$

Now suppose f_1, f_2 are both solutions. Then $g = f_1 - f_2 \in V_S$ is a solution of the difference equation $(-1)^{|S|}(\prod_{\alpha\in S} D_\alpha)g = 0$. If $\lambda \in \mathrm{supp}(g)$ is maximal with respect to the partial ordering $\leqq_S$, then $(-1)^{|S|}(\prod_{\alpha\in S} D_\alpha)g(\lambda) = g(\lambda)$. Hence supp(g) is empty. □

Remark: For $\Lambda = \mathbb{Z}$ and $S = \mathbb{N}$ the function p_S is the classical partition function. For Λ the weight lattice Λ_w and S the set of positive roots Δ_K^+ of the pair (K,T) the function p_S was introduced by B. Kostant [14]. Because we will study the restriction of an irreducible representation of K to a closed connected subgroup L of K containing T, we need the function p_S for S some subset of Δ_K^+.

3.3 Asymptotic partition functions.

Let Λ be a lattice in a Euclidean space E and S a subset of Λ contained in a half-space. We assume that S is finite, $s = |S|$ and $k = \mathrm{rank}(S)$. Fix a numbering $\{\alpha_1,\dots,\alpha_s\}$ for the elements of S. Let $\mathbb{R}^s$ be a Euclidean space with standard basis $\{e_1,\dots,e_s\}$ and denote by $\mathbb{Z}^s$ the integral lattice in $\mathbb{R}^s$. The linear map $A_S : \mathbb{R}^s \to E$ is defined by

$$A_S(\sum_{i=1}^{s} x_i e_i) = \sum_{i=1}^{s} x_i \alpha_i$$

Because $\mathrm{Ker}(A_S) \cap \mathbb{Z}^s$ is a lattice of rank equal to the dimension of $\mathrm{Ker}(A_S)$, the Euclidean measure on $\mathrm{Ker}(A_S)$ can be normalized with respect to this lattice, i.e. by taking the volume of a fundamental bloc equal to 1. By translation we get for each $\lambda \in E$ a well-defined measure on $A_S^{-1}(\lambda)$, which we denote by vol_S.

If we put $(\mathbb{R}^+)^s = \{\sum_{i=1}^{s} x_i e_i : x_i \geqq 0 \text{ for all } i\}$, then $A_S^{-1}(\lambda) \cap (\mathbb{R}^+)^s$ is a convex polytope of dimension less than or equal to $(s-k)$. The function $P_S : E \to \mathbb{R}$ is defined by

$$P_S(\lambda) = \mathrm{vol}_S[A_S^{-1}(\lambda) \cap (\mathbb{R}^+)^s]$$

We call P_S the asymptotic partition function of the set S. For $X \subset \mathbb{R}$ and $T \subset S$ we write X.T for $\{\sum_{i=1}^{s} x_i\alpha_i : x_i \in X$ for all i for which $\alpha_i \in T$ and $x_i = 0$ if $\alpha_i \notin T\}$.

Lemma 3.4 a. $\mathrm{supp}(P_S) \subset \mathbb{R}^+.S$

b. $P_S > 0$ on the interior of the cone $\mathbb{R}^+.S$

c. $P_S(t.\lambda) = t^{s-k} P_S(\lambda)$ for $\lambda \in E$, $t > 0$

Proof:

a. Clearly $A_S^{-1}(\lambda) \cap (\mathbb{R}^+)^s$ is non empty if and only if $\lambda \in A_S((\mathbb{R}^+)^s) = \mathbb{R}^+.S$.

b. Because A_S maps the interior of $(\mathbb{R}^+)^s$ onto the interior of the cone $\mathbb{R}^+.S$, the polytope $A_S^{-1}(\lambda) \cap (\mathbb{R}^+)^s$ has non empty interior relative to $A_S^{-1}(\lambda)$ for λ in the interior of the cone $\mathbb{R}^+.S$.

c. This follows because $A_S^{-1}(t.\lambda) \cap (\mathbb{R}^+)^s = t\{A_S^{-1}(\lambda) \cap (\mathbb{R}^+)^s\}$ for $\lambda \in E$, $t > 0$. □

In order to get more insight into the function P_S we will use induction on s. Choose $\alpha \in S$ and let $T = S \setminus \{\alpha\}$. Then $\alpha = \alpha_j$ for some j, and so we identify $\mathbb{R}^{s-1}$ with $\{\sum_{i=1}^{s} x_i e_i \in \mathbb{R}^s : x_j = 0\}$. There are two possibilities.

Case 1: rank(T) = k-1

Then every $\lambda \in \mathbb{R}^+.S$ can be written uniquely in the form $\lambda = \mu + t\alpha$ with $\mu \in \mathbb{R}^+.T$ and $t \in \mathbb{R}^+$. The projection $q : \mathbb{R}^s \to \mathbb{R}^{s-1}$, defined by $q(\sum_{i=1}^{s} x_i e_i) = \sum_{i \neq j} x_i e_i$, is a bijection from $A_S^{-1}(\lambda) \cap (\mathbb{R}^+)^s$ onto $A_T^{-1}(\mu) \cap (\mathbb{R}^+)^{s-1}$. Moreover, the push-forward under q of vol_S is equal to vol_T. Hence $P_S(\lambda) = P_T(\mu)$.

Case 2: rank(T) = k

Take $\lambda \in \mathbb{R}^+.S$. Define $a(\lambda), b(\lambda) \in \mathbb{R}^+$ by

$$a(\lambda) = \inf\{t \in \mathbb{R}^+ : \lambda - t\alpha \in \mathbb{R}^+.T\}$$
$$b(\lambda) = \sup\{t \in \mathbb{R}^+ : \lambda - t\alpha \in \mathbb{R}^+.T\}$$

Clearly $a(\lambda)$ and $b(\lambda)$ are continuous piece-wise linear functions on $\mathbb{R}^+.S$. The following formula is obvious

$$A_S^{-1}(\lambda) \cap (\mathbb{R}^+)^s = \bigcup_{t \geq 0} \{[A_T^{-1}(\lambda - t\alpha) \cap (\mathbb{R}^+)^{s-1}] + te_j\}$$

Hence we have

$$P_S(\lambda) = \frac{1}{n_\alpha} \int_{a(\lambda)}^{b(\lambda)} P_T(\lambda - t\alpha)\, dt$$

where $n_\alpha \in \mathbb{N}$ is the smallest positive integer such that $n_\alpha.\alpha \in \mathbb{Z}.T$.

Lemma 3.5 The function P_S is continuous on $\mathbb{R}^+.S$. Moreover, if we divide the cone $\mathbb{R}^+.S$ into smaller cones by the hyperplanes $\mathbb{R}.T$, where T ranges over all subsets of S of rank $(k-1)$, then P_S is a polynomial function of degree $(s-k)$ on each of these smaller cones.

Proof: Use induction on s. Choose $\alpha \in S$ and let $T = S \setminus \{\alpha\}$. By induction the lemma is true for P_T. If rank(T) = k-1, then the results of the lemma for P_T extend to P_S in a trivial way. So we assume that rank(T) = k. Using the integral formula for P_S the continuity of P_S on $\mathbb{R}^+.S$ follows immediatly. In fact, a primitive function of a piece-wise polynomial function is again piece-wise polynomial. Because the lower and upper bound $a(\lambda)$ and $b(\lambda)$ are piece-wise linear on $\mathbb{R}^+.S$, the lemma follows. □

Suppose $S = S_1 \cup S_2$ is a disjoint union of two non empty subsets such that $\mathrm{rank}(S_1) + \mathrm{rank}(S_2) = \mathrm{rank}(S)$. Then each $\lambda \in \mathbb{R}^+.S$ can be written uniquely in the form $\lambda = \lambda_1 + \lambda_2$ with $\lambda_1 \in \mathbb{R}^+.S_1$ and $\lambda_2 \in \mathbb{R}^+.S_2$ It is easy to verify that $P_S(\lambda) = P_{S_1}(\lambda_1).P_{S_2}(\lambda_2)$. We say that P_S is irreducible if there does not exist such a splitting. Otherwise P_S is called reducible. If $S = \{\alpha\}$ consists of one single element, then P_S is equal to the Heaviside- function, i.e. $P_S(t\alpha) = 1$ for $t \geq 0$ and $P_S(t\alpha) = 0$ for $t < 0$. Clearly the continuity fails for $t = 0$. On the other hand, if P_S is irreducible and $s \geq 2$, then this cannot happen.

Lemma 3.6 If $\mathrm{rank}(S \setminus \{\alpha\}) = \mathrm{rank}(S)$ for each $\alpha \in S$, then P_S is continuous on $\mathbb{R}.S$.

Proof: By Lemma 3.4 and 3.5 it suffices to prove that $P_S = 0$ on the boundary of $\mathbb{R}^+.S$. Fix a boundary point $\lambda \in \mathbb{R}^+.S$. Then there exists $\alpha \in S$ such that $\lambda + t\alpha \notin \mathbb{R}^+.S$ for $t < 0$. If we put $T = S \setminus \{\alpha\}$, then by assumption $\mathrm{rank}(T) = s$. Applying the integral formula for P_S we get $P_S(\lambda) = 0$, because $a(\lambda) = b(\lambda) = 0$. □

Lemma 3.7 There exists a constant $C > 0$ such that

$$P_S(\lambda) \leq C(1+|\lambda|)^{s-k}$$

Proof: The lemma follows immediately from Lemma 3.5 . □

Denote by $C_c^\infty(E)$ the space of smooth functions on E with compact support. Sometimes we will consider P_S also as a distribution on E in the following way

$$<P_S,f> = \int_{\mathbb{R}.S} P_S(\lambda)\, f(\lambda)\, d_S\lambda$$

for $f \in C_c^\infty(E)$, where the measure $d_S\lambda$ on $\mathbb{R}.S$ is normalized with respect to the lattice $\mathbb{Z}.S$. In particular, for S the empty set P_S is the δ - function at the origin. By Lemma 3.7 P_S is a tempered distribution on E. The next lemma has to be considered as an equality of distributions.

Lemma 3.8 $\left(\prod_{\alpha\in S\setminus T} \frac{\partial}{\partial\alpha} \right) P_S = P_T$ for $T \subset S$

Proof: By induction it suffices to prove the lemma for $T = S \setminus \{\alpha\}$ for some $\alpha \in S$. If $\mathrm{rank}(T) = k-1$, then the lemma follows because the derivative of the Heaviside-function is equal to the δ-function. So we assume $\mathrm{rank}(T) = k$. Using the integral formula for P_S we get for $f \in C_c^\infty(E)$

$$\begin{aligned} <\frac{\partial}{\partial\alpha}P_S,f> &= -\int_{\mathbb{R}.S} P_S(\lambda)\, \frac{\partial f}{\partial\alpha}(\lambda)\, d_S\lambda \\ &= -\frac{1}{n_\alpha}\int_{\mathbb{R}.S}\int_0^\infty P_T(\lambda-t\alpha)\, \frac{\partial f}{\partial\alpha}(\lambda)\, dt\, d_S\lambda \\ &= -\frac{1}{n_\alpha}\int_{\mathbb{R}.S}\int_0^\infty P_T(\lambda)\, \frac{\partial f}{\partial\alpha}(\lambda+t\alpha)\, dt\, d_S\lambda \\ &= \frac{1}{n_\alpha}\int_{\mathbb{R}.S} P_T(\lambda)\, f(\lambda)\, d_S\lambda \\ &= \int_{\mathbb{R}.T} P_T(\lambda)\, f(\lambda)\, d_T\lambda \\ &= <P_T,f> \end{aligned}$$

□

The next lemma justifies the name asymptotic partion function.

Lemma 3.9 There exists a constant $C > 0$ such that for $\lambda \in \mathbb{Z}.S$

$$|p_S(\lambda) - P_S(\lambda)| \leq C(1+|\lambda|)^{s-k-1}$$

Proof: If we denote by $\mathbb{Z}^d$ the standard lattice in $\mathbb{R}^d$, then we have for a bounded subset D of $\mathbb{R}^d$

$$\#(\mathbb{Z}^d \cap D) \leq \mathrm{vol}\,\{\mu\in\mathbb{R}^d : d(\mu,D) \leq \tfrac{1}{2}\sqrt{d}\}$$

and by taking complements in a suitable cube around D

$$\#(\mathbb{Z}^d \cap D) \geq \mathrm{vol}\,\{\mu\in\mathbb{R}^d : d(\mu,\mathbb{R}^d \setminus D) > \tfrac{1}{2}\sqrt{d}\}$$

Hence

$$|\#(\mathbb{Z}^d \cap D) - \mathrm{vol}(D)| \leq \mathrm{vol}\,\{\mu\in\mathbb{R}^d : d(\mu,\partial D) \leq \tfrac{1}{2}\sqrt{d}\}$$

We apply this to the situation $d = s-k$, with $A_S^{-1}(\lambda)$ in stead of $\mathbb{R}^d$, $A_S^{-1}(\lambda) \cap \mathbb{Z}^s$ in stead of $\mathbb{Z}^d$ and $D = A_S^{-1}(\lambda) \cap (\mathbb{R}^+)^s$ for some $\lambda \in \mathbb{Z}.S$. Clearly D is a convex polytope and each proper face of D is of the form $A_T^{-1}(\lambda) \cap (\mathbb{R}^+)^t$, where T is a subset of S with $t = |T|$ and $|T| - \mathrm{rank}(T) \leq s-k-1$. Hence the lemma follows from Lemma 3.7 . □

Lemma 3.10 For $\alpha \in S$ we have $p_S(\lambda-\alpha) = p_S(\lambda) - p_{S\setminus\{\alpha\}}(\lambda)$.

Proof:
$$\begin{aligned} p_S(\lambda) &= \#\{\textstyle\sum_{\beta\in S} n_\beta e_\beta : n_\beta \in \mathbb{Z}^+ , \sum_{\beta\in S} n_\beta\beta = \lambda\} \\ &= \#\{\textstyle\sum_{\beta\in S} n_\beta e_\beta : n_\beta \in \mathbb{Z}^+ , n_\alpha = 0, \sum_{\beta\in S} n_\beta\beta = \lambda\} + \\ &\quad \#\{\textstyle\sum_{\beta\in S} n_\beta e_\beta : n_\beta \in \mathbb{Z}^+ , n_\alpha \geq 1, \sum_{\beta\in S} n_\beta\beta = \lambda\} \\ &= \#\{\textstyle\sum_{\beta\in S\setminus\{\alpha\}} n_\beta e_\beta : n_\beta \in \mathbb{Z}^+, \sum_{\beta\in S\setminus\{\alpha\}} n_\beta\beta = \lambda\} + \\ &\quad \#\{e_\alpha + \textstyle\sum_{\beta\in S} n_\beta e_\beta : n_\beta \in \mathbb{Z}^+, \sum_{\beta\in S} n_\beta\beta = \lambda-\alpha\} \\ &= p_S(\lambda-\alpha) + p_{S\setminus\{\alpha\}}(\lambda) . \quad □ \end{aligned}$$

Lemma 3.11 Suppose $\mathrm{rank}(S\setminus\{\alpha\}) = \mathrm{rank}(S)$ for each $\alpha \in S$, and let $\mu \in \mathbb{Z}.S$. Then there exists a constant $C > 0$, depending on μ, such that

$$|p_S(\lambda-\mu) - P_S(\lambda)| \leq C(1+|\lambda|)^{s-k-1}$$

Proof: Write μ in the form $\mu = \sum_{\alpha\in S} m_\alpha\alpha$ with $m_\alpha \in \mathbb{Z}$, such that $O(\mu) = \sum_{\alpha\in S} |m_\alpha|$ is minimal. Now we prove the lemma by induction on $O(\mu)$. If $O(\mu) = 0$, then $\mu = 0$ and the lemma follows from Lemma 3.9 . So, we assume $O(\mu) \geq 1$. Choose $\alpha \in S$ with $m_\alpha \neq 0$, say $m_\alpha \geq 1$. If we put $\nu = \mu-\alpha$, then $O(\nu) = O(\mu) - 1$. Now we have for $\lambda \in \mathbb{Z}.S$

$$|p_S(\lambda-\mu) - P_S(\lambda)| = |p_S(\lambda-\nu-\alpha) - P_S(\lambda)| \leq$$

$$|p_S(\lambda-\nu-\alpha) - p_S(\lambda-\nu)| + |p_S(\lambda-\nu) - P_S(\lambda)| .$$

For the first term we get

$$\begin{aligned} |p_S(\lambda-\nu-\alpha) - p_S(\lambda-\nu)| &= |p_{S\setminus\{\alpha\}}(\lambda-\nu)| \leq \\ |p_{S\setminus\{\alpha\}}(\lambda-\nu) - P_{S\setminus\{\alpha\}}(\lambda-\nu)| &+ |P_{S\setminus\{\alpha\}}(\lambda-\nu)| \leq \\ C_1(1+|\lambda-\nu|)^{s-k-2} &+ C_2(1+|\lambda-\nu|)^{s-k-1} \leq \\ C_3(1+|\lambda|)^{s-k-1} &\quad \text{for some } C_1, C_2, C_3 \in \mathbb{R}^+ . \end{aligned}$$

Using the induction hypothesis, we see that the second term is bounded by $C_4(1+|\lambda|)^{s-k-1}$ for some $C_4 \in \mathbb{R}^+$. The lemma follows by taking $C = C_3 + C_4$. □

3.4 Asymptotic behaviour of multiplicities.

We use the notation of section 1. The following lemma is an easy consequence of Kostant's multiplicity formula [14].

Lemma 3.12 Let $\delta_K = \frac{1}{2}\sum_{\alpha\in\Delta_K^+} \alpha$ and $p_{\Delta_K^+\setminus\Delta_L^+}$ the partition function of of the set $\Delta_K^+\setminus\Delta_L^+$. Then we have for $\lambda \in C_K^+ \cap \Lambda_w$ and $\mu \in C_L^+$

$$m_\lambda^{K,L}(\mu) = \sum_{w\in W_K} \det(w)\, p_{\Delta_K^+\setminus\Delta_L^+}(w(\lambda+\delta_K) - (\mu+\delta_K))$$

Moreover, if we extend $m_\lambda^{K,L}(\mu)$ to all $\lambda,\mu \in \Lambda_w$ by means of this formula, $m_{w(\lambda+\delta_K)-\delta_K}^{K,L}(\mu) = \det(w)\, m_\lambda^{K,L}(\mu)$ and $m_\lambda^{K,L}(v(\mu+\delta_L)-\delta_L) = \det(v)\, m_\lambda^{K,L}(\mu)$ for all $w \in W_K$, $v \in W_L$ and $\lambda,\mu \in \Lambda_w$.

Proof: By Kostant's multiplicity formula we have for $\mu \in C_L^+ \cap \Lambda_w$ and $\nu \in \Lambda_w$

$$m_\mu^{L,T}(\nu) = \sum_{w \in W_L} \det(w)\, p_{\Delta_L^+}(w(\mu+\delta_L) - (\nu+\delta_L))$$

Because $m_\lambda^{K,T} = \sum_{\mu \in C_L^+ \cap \Lambda_w} m_\lambda^{K,L}(\mu)\, m_\mu^{L,T}$ as functions on Λ_w, we find by applying on both sides the difference operator $(-1)^{|\Delta_L^+|} \prod_{\alpha \in \Delta_L^+} D_\alpha$ that

$$\sum_{w \in W_K} \det(w)\, p_{\Delta_K^+ \setminus \Delta_L^+}(w(\lambda+\delta_K) - (\nu+\delta_K)) =$$

$$\sum_{\mu \in C_L^+ \cap \Lambda_w} m_\lambda^{K,L}(\mu) \sum_{w \in W_L} \det(w)\, \varepsilon_{w(\mu+\delta_L)-\delta_L}(\nu)$$

The assertions of the lemma follow easily from this formula. □

Lemma 3.13 If Δ_K is an irreducible root system and $\Delta_L \subsetneqq \Delta_K$ a root subsystem, then $\mathbb{Z}.(\Delta_K^+ \setminus \Delta_L^+)$ is equal to the root lattice Λ_r.

Proof: Let $\Delta_1 = \mathbb{Z}.(\Delta_K \setminus \Delta_L) \cap \Delta_K$ and $\Delta_2 = \{\alpha \in \Delta_K : (\alpha,\beta) = 0$ for all $\beta \in \Delta_K \setminus \Delta_L\}$. Clearly Δ_1 and Δ_2 are root subsystems of Δ_K. We want to prove that $\Delta_K = \Delta_1 \cup \Delta_2$. Because Δ_K is irreducible and Δ_1 is non empty by assumption, this shows that Δ_2 is empty, which proves the lemma. If $\alpha \in \Delta_K \setminus \Delta_L$, then $\alpha \in \Delta_1$ and we are done. If $\alpha \in \Delta_L$ and $\alpha \notin \Delta_2$, then there exists $\beta \in \Delta_K \setminus \Delta_L$ such that $(\alpha,\beta) \neq 0$, say $(\alpha,\beta) > 0$. Hence $(\alpha-\beta) \in \Delta_K = (\Delta_K \setminus \Delta_L) \cup \Delta_L$. If $(\alpha-\beta) \in \Delta_L$, then $\beta = (\beta-\alpha) + \alpha \in \Delta_L$, which gives a contradiction. Therefore $(\alpha-\beta) \in \Delta_K \setminus \Delta_L$, and so $\alpha = (\alpha-\beta) + \beta \in \Delta_1$. □

Lemma 3.14 Let Δ_K be an irreducible root system and $\Delta_L \subsetneqq \Delta_K$ a root subsystem. If the pair (Δ_K,Δ_L) is of type (A_l,A_{l-1}) or (B_l,D_l), then $\Delta_K^+ \setminus \Delta_L^+$ consists of l linear independent roots. In all other cases we have $\operatorname{rank}(\Delta_K^+ \setminus \Delta_L^+ \setminus \{\alpha\}) = \operatorname{rank}(\Delta_K^+ \setminus \Delta_L^+)$ for each root $\alpha \in \Delta_K^+ \setminus \Delta_L^+$.

Proof: The first assertion is easy to check. In order to prove the second assertion we may assume that $\Delta_L \subsetneqq \Delta_K$ is a maximal root subsystem. However, the pairs (Δ_K,Δ_L) with Δ_K an irreducible root system and $\Delta_L \subsetneqq \Delta_K$ a maximal root subsystem have been classified by A. Borel and J. de Siebenthal [2]. Now the proof of the lemma follows by checking their list case by case. Although this verification is a bit of work, we leave it out of the text because it does not give much insight into the problem. □

Let Δ_K be an irreducible root system and $\Delta_L \subsetneqq \Delta_K$ a root subsystem. We assume that (Δ_K,Δ_L) is not equal to (A_l,A_{l-1}) or (B_l,D_l). For $\lambda \in \sqrt{-1}.t^*$ we define the function $M_\lambda^{K,L} : \sqrt{-1}.t^* \to \mathbb{R}$ by

$$M_\lambda^{K,L}(\mu) = \sum_{w \in W_K} \det(w)\, P_{\Delta_K^+ \setminus \Delta_L^+}(w\lambda - \mu)$$

By Lemma 3.5 and 3.6 the function $M_\lambda^{K,L}$ is a continuous piece-wise polynomial function on $\sqrt{-1}.t^*$. Moreover $M_{t\lambda}^{K,L}(t\mu) = t^r M_\lambda^{K,L}(\mu)$ for $t > 0$ and $r = |\Delta_K^+ \setminus \Delta_L^+| - \operatorname{rank}(\Delta_K^+ \setminus \Delta_L^+)$.

Lemma 3.15 There exists a constant $C > 0$ such that for $\lambda \in \Lambda_w$ and $\mu \in \lambda + \Lambda_r$ we have

$$| m_\lambda^{K,L}(\mu) - M_\lambda^{K,L}(\mu) | \leq C(1 + |\lambda|)^{r-1}$$

Proof: By Lemma 3.12 we have

$$| m_\lambda^{K,L}(\mu) - M_\lambda^{K,L}(\mu) | \leq$$

$$\sum_{w \in W_K} | P_{\Delta_K^+ \setminus \Delta_L^+}(w\lambda - \mu + w\delta_K - \delta_K) - P_{\Delta_K^+ \setminus \Delta_L^+}(w\lambda - \mu) |$$

Because $\delta_K - w\delta_K$ is the sum of all roots in $\Delta_K^+ \cap w\Delta_K^-$ we see from Lemma 3.13 that $\delta_K - w\delta_K \in \Lambda_r$. So the conditions of Lemma 3.11 are fulfilled and we get

$$| m_\lambda^{K,L}(\mu) - M_\lambda^{K,L}(\mu) | \leq \sum_{w \in W_K} C_w(1+|w\lambda-\mu|)^{r-1}$$

$$\leq C_1(1+|\lambda|+|\mu|)^{r-1}$$

for some $C_1 > 0$.

For $|\mu| \leqq |\lambda| + 2|\delta_L|$ we have

$$|m_\lambda^{K,L}(\mu) - M_\lambda^{K,L}(\mu)| \leqq C_1(1+2|\lambda|+2|\delta_L|)^{r-1}$$
$$\leqq C\,(1+|\lambda|)^{r-1}$$

for some $C > 0$.

For $|\mu| > |\lambda| + 2|\delta_L|$ it is easy to see that $m_\lambda^{K,L}(\mu) = 0$, which implies that

$$|M_\lambda^{K,L}(\mu)| = n^{-r}.|M_{n\lambda}^{K,L}(n\mu)| \leqq C\, n^{-r}(1+|n\lambda|+|n\mu|)^{r-1}$$

for all $n \in \mathbb{N}$. Hence $M_\lambda^{K,L}(\mu) = 0$ for $|\mu| > |\lambda| + 2|\delta_L|$. This proves the lemma. □

<u>Lemma 3.16</u> For all $\lambda,\mu \in \sqrt{-1}.t^*$ we have

a. $M_{w\lambda}^{K,L}(\mu) = \det(w)\, M_\lambda^{K,L}(\mu)$ for $w \in W_K$.

b. $M_\lambda^{K,L}(v\mu) = \det(v)\, M_\lambda^{K,L}(\mu)$ for $v \in W_L$.

c. $M_\lambda^{K,T}(w\mu) = M_\lambda^{K,T}(\mu)$ for $w \in W_K$.

Proof: Clearly, part a. is trivial from the definition of $M_\lambda^{K,L}$.

We shall only prove part b. because the proof of part c. goes exactly along the same lines.

For $\lambda \in \Lambda_w$ and $\mu \in \lambda + \Lambda_r$ we have

$$\begin{aligned} M_\lambda^{K,L}(v\mu) &= \lim_{n\to\infty} M_\lambda^{K,L}(v\mu + \tfrac{1}{n}(v\delta_L - \delta_L)) \\ &= \lim_{n\to\infty} n^{-r}.M_{n\lambda}^{K,L}(v(n\mu+\delta_L) - \delta_L) \\ &= \lim_{n\to\infty} n^{-r}.m_{n\lambda}^{K,L}(v(n\mu+\delta_L) - \delta_L) \\ &= \det(v) \lim_{n\to\infty} n^{-r}.m_{n\lambda}^{K,L}(n\mu) \\ &= \det(v) \lim_{n\to\infty} n^{-r}.M_{n\lambda}^{K,L}(n\mu) \\ &= \det(v)\, M_\lambda^{K,L}(\mu) \end{aligned}$$

Because $M_{t\lambda}^{K,L}(t\mu) = t^r.M_\lambda^{K,L}(\mu)$ for all $t > 0$ and $[\Lambda_w:\Lambda_r] < \infty$, we get $M_\lambda^{K,L}(v\mu) = \det(v)\, M_\lambda^{K,L}(\mu)$ for all $\lambda,\mu \in \mathbb{Q}.\Lambda_w$, and because $M_\lambda^{K,L}(\mu)$ is continuous in λ and μ the equality holds for all $\lambda,\mu \in \sqrt{-1}.t^*$. □

<u>Corollary</u>: Suppose λ is singular for W_K, i.e. the stabilizer W_K^λ of λ in W_K is non trivial. Then $M_\lambda^{K,L}(\mu) = 0$ for all $\mu \in \sqrt{-1}.t^*$.

Proof: It is well-known that W_K^λ is a subgroup of W_K generated by reflections. Hence the corollary follows from Lemma 3.16 . □

Now suppose (Δ_K,Δ_L) is equal to (A_1,A_{1-1}) or (B_1,D_1). Because $|\Delta_K^+ \setminus \Delta_L^+| = \mathrm{rank}(\Delta_K^+ \setminus \Delta_L^+)$, we have

$$P_{\Delta_K^+\setminus\Delta_L^+}(\mu) = \begin{cases} 1 & \text{for } \mu \in \mathbb{Z}^+.(\Delta_K^+ \setminus \Delta_L^+) \\ 0 & \text{for } \mu \in \mathbb{Z}^+.(\Delta_K^+ \setminus \Delta_L^+) \end{cases}$$

and

$$P_{\Delta_K^+\setminus\Delta_L^+}(\mu) = \begin{cases} 1 & \text{for } \mu \in \mathbb{R}^+.(\Delta_K^+ \setminus \Delta_L^+) \\ 0 & \text{for } \mu \in \mathbb{R}^+.(\Delta_K^+ \setminus \Delta_L^+) \end{cases}$$

We define the function $M_\lambda^{K,L} : \sqrt{-1}.t^* \to \mathbb{R}$ in this case by

$$M_\lambda^{K,L}(\mu) = \lim_{\varepsilon\downarrow 0} \sum_{w\in W_K} \det(w)\, P_{\Delta_K^+\setminus\Delta_L^+}(w\lambda-\mu+\varepsilon(w\delta_K-\delta_K))$$

Clearly we have $M_{t\lambda}^{K,L}(t\mu) = t^r.M_\lambda^{K,L}(\mu)$ for all $t > 0$.

<u>Lemma 3.17</u> Suppose (Δ_K,Δ_L) is of type (A_1,A_{1-1}) or (B_1,D_1). For $\lambda \in C_K^+ \cap \Lambda_w$ and $\mu \in C_L^+ \cap (\lambda+\Lambda_r)$ we have

$$m_\lambda^{K,L}(\mu) = M_\lambda^{K,L}(\mu)$$

Proof: We will give a proof of this lemma in section 3 of chapter 4. □

So we have proved

Theorem 4. Let Δ_K be an irreducible root system and $\Delta_L \subsetneq \Delta_K$ a root subsystem. Then there exists a constant $C > 0$ such that for $\lambda \in C_K^+ \cap \Lambda_w$ and $\mu \in C_L^+ \cap (\lambda + \Lambda_r)$

$$| m_\lambda^{K,L}(\mu) - M_\lambda^{K,L}(\mu) | \leqq C(1+|\lambda|)^{r-1}$$

where $r = |\Delta_K^+ \setminus \Delta_L^+| - \mathrm{rank}(\Delta_K^+ \setminus \Delta_L^+)$. The function $M_\lambda^{K,L}$ is a piece-wise polynomial on $\sqrt{-1}.t^*$ and satisfies the relation $M_{t\lambda}^{K,L}(t\mu) = t^r.M_\lambda^{K,L}(\mu)$ for all $t > 0$.

From now on Δ_K is an irreducible root system and $\Delta_L \subsetneq \Delta_K$ a root subsystem. We denote by $d\mu$ the measure on $\sqrt{-1}.t^*$ normalized with respect to the root lattice. We will consider the function $M_\lambda^{K,L}$ also as a distribution on $\sqrt{-1}.t^*$ by

$$< M_\lambda^{K,L}, f > \;=\; \int_{\sqrt{-1}.t^*} M_\lambda^{K,L}(\mu)\, f(\mu)\, d\mu$$

for $f \in C_c^\infty(\sqrt{-1}.t^*)$.

Lemma 3.18 We have the equality of distributions

$$(-1)^{|\Delta_K^+ \setminus \Delta_L^+|} \Big(\prod_{\alpha \in \Delta_K^+ \setminus \Delta_L^+} \frac{\partial}{\partial \alpha} \Big) M_\lambda^{K,L} = \sum_{w \in W_K} \det(w)\, \delta_{w\lambda}$$

Proof: The proof follows from Lemma 3.8 and 3.13 and the definition of $M_\lambda^{K,L}$. □

The functions π_K and π_L on t are defined by

$$\pi_K(H) = \prod_{\alpha \in \Delta_K^+} \alpha(H)$$

$$\pi_L(H) = \prod_{\alpha \in \Delta_L^+} \alpha(H)$$

for $H \in t$. Clearly, π_K is a skew-W_K-invariant and π_L a skew-W_L-invariant polynomial function.

Lemma 3.19 For a W_L-regular point $H \in t$ we have

$$\frac{\pi_K(H)}{\pi_L(H)} \int_{\sqrt{-1}.t^*} M_\lambda^{K,L}(\mu)\, e^{\mu(H)}\, d\mu = \sum_{w \in W_K} \det(w)\, e^{w\lambda(H)}$$

Proof: The proof follows from Lemma 3.18 by Fourier transformation. □

3.5 The push-forward of a measure.

Let M be an oriented Riemannian manifold of dimension m and dm the volume form on M. Let $p: M \to \mathbb{R}^n$ be a proper smooth map. We assume that there exists an open dense subset M' of M such that $p^{-1}(p(M')) = M'$ and $p|_{M'}$ is a submersion. We define a function $D : \mathbb{R}^n \to \mathbb{R}^+$ as follows:
For $x \in \mathbb{R}^n \setminus p(M')$ we put $D(x) = 0$. If $x \in p(M')$, then $p^{-1}(x)$ is a compact submanifold of M of dimension (m-n). Due to the Riemannian structure $p^{-1}(x)$ carries a natural measure $d_x(m)$. Now we put

$$D(x) = \int_{p^{-1}(x)} (Jp)^{-1}(m)\, d_x(m)$$

where for $m \in p^{-1}(x)$ the Jacobian Jp is defined by $Jp(m) = \det(dp(m)^\perp)$, where $dp(m)^\perp$ denotes the restriction of $dp(m)$ to $T_m(p^{-1}(x))^\perp$. It is clear that the support of D is equal to $\overline{p(M')} = p(M)$.

Lemma 3.20 For $f \in C_c(\mathbb{R}^n)$ we have

$$\int_M f(p(m))\, dm = \int_{\mathbb{R}^n} f(x)\, D(x)\, dx$$

In particular, D is a locally summable function on $\mathbb{R}^n$.

Proof: The lemma follows in a standard way by applying the Jacobi-substitution theorem. □

We want to apply this lemma in the situation that p is the orthogonal projection p_L from k onto ℓ and M is the K-orbit $Ad(K)H_0$ through some $H_0 \in t$. Unfortunately, in general p_L fails to be a submersion. However we have

Lemma 3.21 Suppose k is a compact simple Lie algebra, $\ell \subsetneq k$ a subalgebra of the same rank, $t \subset \ell$ a maximal torus and H_0 a W_K-regular point of t. Then there exists a dense open subset M' of $M = Ad(K)H_0$ such that $p_L^{-1}(p_L(M')) = M'$ and $p_L : M' \to \ell$ is a submersion.

Proof: Because p_L commutes with $Ad(l)$ for each $l \in L$, the set $p_L(Ad(K)H_0)$ is Ad(L)-invariant. Hence $p_L(Ad(K)H_0) \cap t$ is W_L-invariant. We say that $Y \in \ell$ is L-regular if the Ad(L)-orbit through Y has maximal dimension. It is well-known that $Y \in t$ is L-regular if and only if Y is W_L-regular, i.e. the W_L-orbit through Y contains exactly $|W_L|$ points. Therefore, the L-regular elements of ℓ form a non-empty Zariski-open subset of ℓ. Because $Ad(K)H_0$ is an irreducible real variety, the set $M'' = \{X \in M: p_L(X) \text{ is L-regular}\}$ is a non-empty Zariski-open subset of M. It is clear that M" is Ad(L)-invariant.

Now we fix $X \in M''$ such that $p_L(X) \in C_L^+$. Then $dp_L(T_X(M))$ is a direct sum of $dp_L(T_X(M)) \cap t$ and $T_{p_L(X)}(Ad(L)p_L(X))$, and therefore $dp_L(T_X(M)) \cap t$ is equal to $dp_T(T_X(M))$. Suppose p_L is not a submersion at X. Then we can choose $H \in t$, $H \neq 0$, and H perpendicular to $dp_T(T_X(M))$. By Lemma 1.2 we see that $X \in Ad(K^H)Ad(w)H_0$ for some $w \in W_K$,

and the convexity theorem implies that $p_L(X) = p_T(X)$ lies in the convex hull of $Ad(W_K^H)Ad(w)H_0$. By Lemma 3.13 and 3.23 it follows that $M''' = \{X \in M'': p_L \text{ is a submersion at } X\}$ is a non-empty Zariski-open subset of M. So we can take $M' = p_L^{-1}(p_L(M) \setminus p_L(M \setminus M'''))$. □

We can choose a Weyl basis $\{Z_\alpha : \alpha \in \Delta_K\}$ for $k_{\mathbb{C}}$ such that

$$k = t \oplus \sum_{\alpha\in\Delta_K^+} \mathbb{R}.E_\alpha \oplus \sum_{\alpha\in\Delta_K^+} \mathbb{R}.F_\alpha$$

where $E_\alpha = \sqrt{\tfrac{1}{2}}.(Z_\alpha + Z_{-\alpha})$ and $F_\alpha = \sqrt{-\tfrac{1}{2}}.(Z_\alpha - Z_{-\alpha})$. We put $H_\alpha = [E_\alpha, F_\alpha]$ for $\alpha \in \Delta_K$. If we take as inner product on k minus the Killing form, then it is easy to see that

$$(H_\alpha, H) = -\sqrt{-1}.\alpha(H) \qquad \text{for } H \in t,\ \alpha \in \Delta_K.$$

The conjugation map $c_L : \ell \to C_L^+$ is defined by

$$c_L(X) = Ad(L)X \cap C_L^+ \qquad \text{for } X \in \ell.$$

It is well-known that c_L is smooth on the L-regular points of ℓ.

Lemma 3.22 Let H_0 be a W_L-regular point of t, and $X = \sum_{\alpha\in\Delta_L^+}(c_\alpha E_\alpha + d_\alpha F_\alpha)$ a point of $\ell \cap k^\perp$. Then we have

$$c_L(H_0 + tX) = H_0 + \frac{t^2}{2} \sum_{\alpha\in\Delta_L^+} \frac{c_\alpha^2 + d_\alpha^2}{(H_0, H_\alpha)} H_\alpha + O(t^3)$$

Proof: Clearly the function $t \to c_L(H_0 + tX)$ is smooth in a neighbourhood of $t = 0$, and $\frac{d}{dt}\{c_L(H_0+tX)\}_{t=0} = 0$.

If we put $Y = \sum_{\alpha\in\Delta_L^+} \frac{d_\alpha E_\alpha - c_\alpha F_\alpha}{(H_0, H_\alpha)} \in \ell$, then one can easily check that

$$[Y, H_0] = -X$$

$$[Y, X] \in \sum_{\alpha\in\Delta_L^+} \frac{c_\alpha^2 + d_\alpha^2}{(H_0, H_\alpha)} H_\alpha + \sum_{\alpha\in\Delta_L^+} (\mathbb{R}.E_\alpha + \mathbb{R}.F_\alpha)$$

Now we get

$$c_L(H_0+tX) = c_L(\mathrm{Ad}(\exp tY)(H_0+tX))$$
$$= c_L(H_0+\tfrac{1}{2}t^2[Y,X]+O(t^3))$$
$$= H_0 + \tfrac{1}{2}t^2 \sum_{\alpha\in\Delta_L^+} \frac{c_\alpha^2+d_\alpha^2}{(H_0,H_\alpha)} H_\alpha + O(t^3). \quad \square$$

Lemma 3.23 Let H_0 be a W_K-regular point of t, and let $X = \sum_{\alpha\in\Delta_K^+\setminus\Delta_L^+}(c_\alpha E_\alpha + d_\alpha F_\alpha)$ be a point of $k \cap \ell^\perp$. Then

$$c_L(p_L(\mathrm{Ad}(\exp tX)H_0)) = H_0 - \tfrac{1}{2}t^2 \sum_{\alpha\in\Delta_K^+\setminus\Delta_L^+}(c_\alpha^2+d_\alpha^2)(H_0,H_\alpha)H_\alpha + O(t^3)$$

Proof: We have

$$p_L(\mathrm{Ad}(\exp tX)H_0) = H_0 + t\,p_L([X,H_0]) + \tfrac{1}{2}t^2 p_L([X,[X,H_0]]) + O(t^3)$$
$$= H_0 + \tfrac{1}{2}t^2 p_L([X,[X,H_0]]) + O(t^3),$$

hence

$$c_L(p_L(\mathrm{Ad}(\exp tX)H_0)) = H_0 + \tfrac{1}{2}t^2 p_T([X,[X,H_0]]) + O(t^3)$$

by Lemma 3.22 , and one easily checks that

$$p_T([X,[X,H_0]]) = -\tfrac{1}{2}t^2 \sum_{\alpha\in\Delta_K^+\setminus\Delta_L^+} (c_\alpha^2+d_\alpha^2)(H_0,H_\alpha)H_\alpha . \quad \square$$

3.6 The functorial property of the orbit method.

We keep to the notation of the preceding sections. We assume that k is simple and $\ell \neq k$. Fix some Ad(K)-invariant inner product (.,.) on $\sqrt{-1}.k^*$. We introduce functions π_K, π_L, d_K and d_L on $\sqrt{-1}.t^*$ by

$$\pi_K(\lambda) = \prod_{\alpha\in\Delta_K^+}(\alpha,\lambda) \qquad \pi_L(\lambda) = \prod_{\alpha\in\Delta_L^+}(\alpha,\lambda)$$

$$d_K(\lambda) = \prod_{\alpha\in\Delta_K^+}\frac{(\alpha,\lambda)}{(\alpha,\delta_K)} \qquad d_L(\lambda) = \prod_{\alpha\in\Delta_L^+}\frac{(\alpha,\lambda)}{(\alpha,\delta_L)}$$

where as before $\delta_K = \frac{1}{2}\sum_{\alpha\in\Delta_K^+}\alpha$ and $\delta_L = \frac{1}{2}\sum_{\alpha\in\Delta_L^+}\alpha$.

We denote by dk and dl the normalized invariant measures on Ad(K) and Ad(L) respectively. If we denote by $d\mu$ the Euclidean measure on $\sqrt{-1}.t^*$ normalized with respect to the root lattice, then it is well-known that the translation invariant measure $d\nu$ on $\sqrt{-1}.\ell^*$ can be so normalized that Weyl's integral formula holds:

$$\int_{\sqrt{-1}.\ell^*} f(\nu)\,d\nu = \frac{1}{|W_L|}\int_{\sqrt{-1}.t^*}\pi_L(\mu)^2\int_{\mathrm{Ad}(L)} f(\mathrm{Ad}(l)\mu)\,dl\,d\mu$$

for all $f \in C_c(\sqrt{-1}.\ell^*)$.

Using Lemma 3.20 and 3.21 there exists a locally summable function $D_\lambda^{K,L} : \sqrt{-1}.\ell^* \to \mathbb{R}$ such that

$$\int_{\mathrm{Ad}(K)} f(p_L(\mathrm{Ad}(k)\lambda))\,dk = \int_{\sqrt{-1}.\ell^*} f(\nu)\,D_\lambda^{K,L}(\nu)\,d\nu$$

for each $f \in C(\sqrt{-1}.\ell^*)$. The function $D_\lambda^{K,L}$ is Ad(L)-invariant and the support of $D_\lambda^{K,L}$ is equal to $p_L(\mathrm{Ad}(K)\lambda)$.

Theorem 5. For $\lambda \in \mathrm{Int}(C_K^+)$ we have for almost all $\mu \in \sqrt{-1}.t^*$

$$d_L(\mu)\,M_\lambda^{K,L}(\mu) = d_K(\lambda)\,\pi_L(\mu)^2\,D_\lambda^{K,L}(\mu)$$

Proof: The proof is based on the following formula of Harish-Chandra [11]

$$\pi_K(H)\,d_K(\lambda)\int_{\mathrm{Ad}(K)} e^{\lambda(\mathrm{Ad}(k)H)}dk = \sum_{w\in W_K}\det(w)\,e^{\lambda(wH)}$$

We have for W_K-regular $H \in t$

$$\int_{\mathrm{Ad}(K)} e^{\lambda(\mathrm{Ad}(k)H)}\,dk = \int_{\sqrt{-1}.\ell^*} D_\lambda^{K,L}(\nu)\,e^{\nu(H)}\,d\nu =$$

$$= \frac{1}{|W_L|} \int_{\sqrt{-1}.t^*} \pi_L(\mu)^2 D_\lambda^{K,L}(\mu) \int_{Ad(L)} e^{\mu(Ad(l)H)}\, dl\, d\mu$$

$$= \frac{1}{|W_L|} \int_{\sqrt{-1}.t^*} \frac{\pi_L(\mu)^2 D_\lambda^{K,L}(\mu)}{d_L(\mu)\ \pi_L(H)} \left\{ \sum_{w\in W_L} \det(w)\, e^{w\mu(H)} \right\} d\mu$$

$$= \int_{\sqrt{-1}.t^*} \frac{\pi_L(\mu)^2 D_\lambda^{K,L}(\mu)}{d_L(\mu)\ \pi_L(H)}\, e^{\mu(H)}\, d\mu \quad ,$$

hence the left hand side of Harish-Chandra's formula becomes

$$d_K(\lambda)\, \frac{\pi_K(H)}{\pi_L(H)} \int_{\sqrt{-1}.t^*} \frac{\pi_L(\mu)^2}{d_L(\mu)}\, D_\lambda^{K,L}(\mu)\, e^{\mu(H)}\, d\mu \quad .$$

On the other hand, by Lemma 3.19, the right hand side of the formula of Harish-Chandra is equal to

$$\frac{\pi_K(H)}{\pi_L(H)} \int_{\sqrt{-1}.t^*} M_\lambda^{K,L}(\mu)\, e^{\mu(H)}\, d\mu \quad .$$

So the theorem follows from Fourier inversion. □

<u>Remark</u>: For L = T the connection between the function $D_\lambda^{K,T}$ and the asymptotic behaviour of the multiplicities $m_\lambda^{K,T}$ has been remarked by V. Guillemin [9].

CHAPTER 4

EXAMPLES

4.1 Introduction.

In this chapter we will have a closer look at the relation between the multiplicity function $m_\lambda^{K,L}$ and the asymptotic multiplicity function $M_\lambda^{K,L}$. In section 2 we prove that

$$\mathrm{supp}(M_\lambda^{K,L}) \cap C_L^+ \subset \bigcap_{w\in W_K, w\lambda\in C_L^+} \{w\lambda + \sum_{\alpha\in\Delta_K\setminus\Delta_L, (\alpha,w\lambda)>0} \mathbb{R}^-.\alpha\} \cap C_L^+$$

In fact, quite often this inclusion seems to be an equality. For example, if L = T this follows from Theorem 1. Indeed, if g is a complex semisimple Lie algebra, then multiplication by $\sqrt{-1}$ intertwines the adjoint action of K on k and on p. However, examples show that the inclusion can also fail to be an equality, e.g. if the pair (K,L) is of type (B_1,D_1) or (G_2,A_2). For 1 = rank(K) = 2 these are the only examples.

In section 3 we consider the case where $m_\lambda^{K,L}$ only has values 0 and ±1. Using the classification table of Borel and de Siebenthal it follows that this can occur only if (K,L) is of type (B_1,D_1) or (A_1,A_{1-1}). These cases are treated in detail.

In section 4 we explain the reduction for (K,L) of type (G_2,A_2). The multiplicity function has an analogous behaviour as the inner multiplicities for A_2. The reason behind this fact seems to be that $\Delta_K \setminus \Delta_L$ is again a root system of type A_2.

In section 5 we conclude this chapter with a discussion on further problems.

4.2 An upper-bound for $m_\lambda^{K,L}$.

The following lemma is a consequence of the Poincaré-Birkhoff-Witt (or PBW) theorem [14].

Lemma 4.1 For $\lambda \in C_K^+$ and $\mu \in C_L^+$ we have

$$m_\lambda^{K,L}(\mu) \leqq p_{\Delta_K^+ \setminus \Delta_L^+}(\lambda-\mu)$$

Proof: Fix a Weyl basis $\{Z_\alpha : \alpha \in \Delta_K^+\}$ for $k_{\mathbb{C}}$. We define nilpotent Lie algebras n_K^+, n_L^+ by

$$n_K^+ = \sum_{\alpha \in \Delta_K^+} \mathbb{C}.Z_\alpha \qquad n_L^+ = \sum_{\alpha \in \Delta_L^+} \mathbb{C}.Z_\alpha$$

For a complex Lie algebra g we denote by $U(g)$ the universal enveloping algebra of g.

Suppose v is a non-zero highest weight vector for n_L^+ in the representation space $V(\lambda,K)$ of $\pi(\lambda,K)$ of weight μ, i.e.

$$d\pi(\lambda,K)(X)v = 0$$

for all $X \in n_L^+$. Then $V' = d\pi(\lambda,K)(U(n_K^+))v$ is a finite dimensional n_K^+-module. Hence, by Engel's theorem there exists $v_0 \in V'$, $v_0 \neq 0$, such that

$$d\pi(\lambda,K)(X)v_0 = 0$$

for all $X \in n_K^+$. Clearly, v_0 is the (up to a constant unique) highest weight vector in $V(\lambda,K)$. Choose $Y \in U(n_K^+)$ such that $d\pi(\lambda,K)(Y)v = v_0$. Fix an ordering for the positive roots so that $\Delta_K^+ \setminus \Delta_L^+ = \{\alpha_1,\ldots,\alpha_s\}$ and $\Delta_L^+ = \{\alpha_{s+1},\ldots,\alpha_r\}$. According to the PBW theorem we can write

$$Y = \sum_{i=1}^{k} Y_{i,1} \otimes Y_{i,2}$$

with $Y_{i,1} = \prod_{j=1}^{s} (Z_{\alpha_j})^{n_{i,j}}$ and $Y_{i,2} \in U(n_L^+)$.
Suppose $Y_{i,2} \in c_i.1 + \sum_{\alpha\in\Delta_L^+} U(n_L^+)Z_\alpha$ and put $Z = \sum_{i=1}^{k} c_i Y_{i,1}$. Then

$$d\pi(\lambda,K)(Y)v = d\pi(\lambda,K)(Z)v = v_0$$

Let V_1 be the subspace of $U(n_K^+)$ with basis $\{ \prod_{j=1}^{s} (Z_{\alpha_j})^{n_{i,j}} : \sum_{j=1}^{s} n_{i,j}\alpha_j = \lambda-\mu \}$. We denote the space of highest weight vectors in $V(\lambda,K)$ for n_L^+ by V_2. Then the map $(Z,v) \longrightarrow d\pi(\lambda,K)(Z)v$ defines a pairing between the vector spaces V_1 and V_2. It follows from what we said above that this pairing is non-singular on V_2. Hence

$$m_\lambda^{K,L}(\mu) = \dim(V_1) \leqq \dim(V_2) = p_{\Delta_K^+\setminus\Delta_L^+}(\lambda-\mu) \qquad \square$$

Lemma 4.2 If $w \in W_K$ such that $wC_K^+ \subset C_L^+$, then we have for all $\lambda \in C_K^+$, $\mu \in C_L^+$

$$m_\lambda^{K,L}(\mu) \leqq p_{w\Delta_K^+\setminus\Delta_L^+}(w\lambda-\mu)$$

Proof: This follows immediatly from Lemma 4.1 if we had chosen the Weyl chamber $wC_K^+ \subset C_L^+$ in stead of C_K^+. $\square$

Corollary 1 For W_K-regular $\lambda \in \Lambda_W$ we have

$$\operatorname{supp}(m_\lambda^{K,L}) \cap C_L^+ \subset \bigcap_{w\in W_K, w\lambda \in C_L^+} \{w\lambda + \sum_{\alpha\in\Delta_K\setminus\Delta_L, (\alpha,w\lambda)>0} \mathbb{Z}^-.\alpha\}$$

Corollary 2 For $\lambda \in \sqrt{-1}.t^*$ we have

$$\operatorname{supp}(M_\lambda^{K,L}) \cap C_L^+ \subset \bigcap_{w\in W_K, w\lambda \in C_L^+} \{w\lambda + \sum_{\alpha\in\Delta_K\setminus\Delta_L, (\alpha,w\lambda)>0} \mathbb{R}^-.\alpha\}$$

4.3 Multiplicity free reduction.

The reduction from K to L is called multiplicity free if $m_\lambda^{K,L}(\mu)$ is either 0 or 1 for all $\lambda \in C_K^+$ and $\mu \in C_L^+$. The following lemma is due to M. Krämer [20].

Lemma 4.3 The following conditions are equivalent:

a. the reduction from K to L is multiplicity free

b. $|\Delta_K^+ \setminus \Delta_L^+| - \mathrm{rank}(\Delta_K^+ \setminus \Delta_L^+) = 0$

Proof:

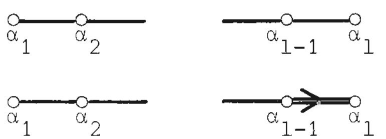
a. ⇒ b. Clearly, if λ is far away from the walls of C_K^+, then $m_\lambda^{K,L}(\mu) = p_{\Delta_K^+\setminus\Delta_L^+}(\lambda-\mu)$ for μ close enough to λ. Hence the partition function $p_{\Delta_K^+\setminus\Delta_L^+}$ has only values 0 and 1. It is easy to see that this can happen only when $|\Delta_K^+ \setminus \Delta_L^+| - \mathrm{rank}(\Delta_K^+ \setminus \Delta_L^+) = 0$.

b. ⇒ a. Suppose $|\Delta_K^+ \setminus \Delta_L^+| - \mathrm{rank}(\Delta_K^+ \setminus \Delta_L^+) = 0$. Then $p_{\Delta_K^+\setminus\Delta_L^+}$ has only values 0 and 1. Hence the lemma follows from Lemma 4.1 . □

A. Borel and J. de Siebenthal [2] have given a list up to local isomorphy of all maximal closed subgroups L of a simple compact connected Lie group K with rank(K) = rank(L). It is easy to check from their table that the conditions of Lemma 4.3 are satisfied only in the cases that the pair (K,L) is of type (A_l, A_{l-1}) or (B_l, D_l). In order to prove Lemma 3.17 we will have a closer look at these cases. The Dynkin diagrams are

A_l :

B_l :



For Δ_K of type A_l we take $\Delta_L = \mathbb{Z}\{\alpha_1,\dots,\alpha_{l-1}\} \cap \Delta_K$, and for Δ_K of type B_l we take $\Delta_L = \mathbb{Z}\{\alpha_1,\dots,\alpha_{l-1},\alpha_{l-1} + 2\alpha_l\} \cap \Delta_K$. If we put

$\beta_i = \alpha_i + \dots + \alpha_l$ for $i=1,\dots,l$, then it is easy to see that $\Delta_K^+ \setminus \Delta_L^+ = \{\beta_1,\dots,\beta_l\}$. Moreover we have for $j=1,\dots,l-1$

$$\frac{2(\beta_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \begin{cases} +1 & \text{for } i=j \\ -1 & \text{for } i=j+1 \\ 0 & \text{for } i \neq j,j+1 \end{cases}$$

Fix a $\lambda \in C_K^+$. Define for $i=1,\dots,l$ the numbers $n_i = n_{\beta_i} \in \mathbb{R}^+$ by

$$n_i = \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}$$

For $\varepsilon > 0$ we define $C(\lambda,\varepsilon)$ by

$$C(\lambda,\varepsilon) = \{\mu \in \sqrt{-1}.\mathfrak{t}^* :\ \mu = \lambda - \sum_{j=1}^{l} x_j\beta_j,\ 0 \leq x_j < n_j + \varepsilon\}$$

Clearly, $C(\lambda,\varepsilon)$ is a parallelepiped with vertices $\lambda - \sum_{\beta\in S}(n_\beta+\varepsilon)\beta$, where S runs over the subsets of $\Delta_K^+ \setminus \Delta_L^+$. In fact, we have

$$\sum_{S\subset\Delta_K^+\setminus\Delta_L^+} (-1)^{|S|}\, p_{\Delta_K^+\setminus\Delta_L^+}(\lambda - \sum_{\beta\in S}(n_\beta+\varepsilon)\beta - \mu) = \begin{cases} 1 & \text{if } \mu \in C(\lambda,\varepsilon) \\ 0 & \text{if } \mu \notin C(\lambda,\varepsilon) \end{cases}$$

We consider the affine action $\mu \to w(\mu+\varepsilon.\delta_K) - \varepsilon.\delta_K$ of W_K on $\sqrt{-1}.\mathfrak{t}^*$ with respect to the origin $-\varepsilon.\delta_K$.

Lemma 4.4 The only vertices of $C(\lambda,\varepsilon)$ which are non-singular for the affine action of W_L on $\sqrt{-1}.\mathfrak{t}^*$ with respect to the origin $-\varepsilon.\delta_K$ are exactly the points

$$\{W_K(\lambda+\varepsilon.\delta_K) - \varepsilon.\delta_K\} \cap C_L^+$$

Proof: Let $\lambda_S = \lambda - \sum_{\beta\in S}(n_\beta+\varepsilon)\beta$ be a vertex of $C(\lambda,\varepsilon)$ for some $S \subset \Delta_K^+ \setminus \Delta_L^+$. For $j=1,\dots,l-1$ we have

$$\frac{2(\lambda_S,\alpha_j)}{(\alpha_j,\alpha_j)} = \begin{cases} n_{j+1} & \text{if } \beta_j,\beta_{j+1} \in S \\ -\varepsilon & \text{if } \beta_j \in S,\ \beta_{j+1} \notin S \\ n_j + n_{j+1} + \varepsilon & \text{if } \beta_j \notin S,\ \beta_{j+1} \in S \\ n_j & \text{if } \beta_j,\beta_{j+1} \notin S \end{cases}$$

Now it is easy to see that the only vertices of $C(\lambda,\varepsilon)$, which are non-singular for the affine action of W_L with respect to $-\varepsilon.\delta_K$, are

for A_1 : $\lambda,\ \lambda-(n_1+\varepsilon)\beta_1,\dots,\ \lambda-(n_1+\varepsilon)\beta_1-\dots-(n_1+\varepsilon)\beta_1$
for B_1 : $\lambda,\ \lambda-(n_1+\varepsilon)\beta_1$

We leave it to the reader to verify that these are indeed exactly the points $\{W_K(\lambda+\varepsilon.\delta_K) - \varepsilon.\delta_K\} \cap C_L^+$. □

Now we put

$$M_{\lambda,\varepsilon}^{K,L}(\mu) = \sum_{w\in W_K} \det(w)\, P_{\Delta_K^+\setminus\Delta_L^+}(w(\lambda+\varepsilon.\delta_K) - (\mu+\varepsilon.\delta_K))$$

Because the satbilizer W_L^μ of μ in W_L is also a group generated by reflections, we have for singular $\mu \in \sqrt{-1}.t^*$

$$\sum_{w\in W_L^\mu} \det(w) = 0$$

So the reader will see immediatly, after having made the above verifications, that Lemma 4.4 yields

$$M_{\lambda,\varepsilon}^{K,L}(\mu) = \sum_{w\in W_L, S\subset\Delta_K^+\setminus\Delta_L^+} \det(w).(-1)^{|S|}.P_{\Delta_K^+\setminus\Delta_L^+}(w(\lambda-\sum_{\beta\in S}(n_\beta+\varepsilon)\beta+\varepsilon.\delta_K)-(\mu+\varepsilon.\delta_K))$$

Hence for $\lambda \in C_K^+$ and $\mu \in C_L^+$ we get

$$M_{\lambda,\varepsilon}^{K,L}(\mu) = \begin{cases} 1 & \text{if } \mu \in C(\lambda,\varepsilon) \\ 0 & \text{if } \mu \notin C(\lambda,\varepsilon) \end{cases}$$

If we put $C(\lambda) = \bigcap_{\varepsilon>0} C(\lambda,\varepsilon) = \{\mu\in\sqrt{-1}.t^*\colon \mu=\lambda-\sum_{j=1}^{l} x_j\beta_j,\ 0\leq x_j\leq n_j\}$, then by choosing $\varepsilon = 1$ we get

<u>Lemma 4.5</u> For $\lambda \in C_K^+ \cap \Lambda_W$ and $\mu \in C_L^+ \cap (\lambda+\Lambda_r)$ we have

$$m_\lambda^{K,L}(\mu) = \begin{cases} 1 & \text{if } \mu \in C(\lambda) \\ 0 & \text{if } \mu \notin C(\lambda) \end{cases}$$

On the other hand, if ε tends to 0, we get

<u>Lemma 4.6</u> For $\lambda \in C_K^+$ and $\mu \in C_L^+$ we have

$$M_\lambda^{K,L}(\mu) = \begin{cases} 1 & \text{if } \mu \in C(\lambda) \\ 0 & \text{if } \mu \notin C(\lambda) \end{cases}$$

and so the proof of Lemma 3.17 is complete.

<u>Remark</u>: Lemma 4.5 is known; we refer to [24].

For l=2 we have drawn pictures of the behaviour of the multiplicity function $m_\lambda^{K,L}$ (figures 1,5). The dots • are the points where $m_\lambda^{K,L}$ is non-zero and the stars ✱ indicate the orbit $W_K(\lambda+\delta_K)-\delta_K$. From these pictures one can read off the set $p_L(Ad(K)\lambda) \cap C_L^+ = supp(M_\lambda^{K,L}) \cap C_L^+$: this is the shaded region in the figures 2 and 6.

From the reduction for (K,L) of type (A_2,A_1) we get the well-known behaviour of the inner multiplicities $m_\lambda^{K,T}(\mu)$ for A_2 (figure 3). Clearly the corresponding function $M_\lambda^{K,T}$ is piece-wise linear (figure 4): $M_\lambda^{K,T}$ is zero outside the convex hull of $W_K.\lambda$ and $M_\lambda^{K,T}$ is constant on the inner triangle. In general this method provides another proof of Theorem 1 for complex groups G.

4.4 The reduction (G_2,A_2).

In this section we take (K,L) of type (G_2,A_2). It has been remarked by Fronsdal [17] that the multiplicity function $m_\lambda^{K,L}$ has an " A_2-behaviour ". We have drawn an example in figure 7. It follows from Lemma 3.3 and Lemma 3.12 that the multiplicity function $m_\lambda^{K,L}$ is the unique compactly supported solution of the difference equation

$$(-1)^{|\Delta_K^+\setminus\Delta_L^+|} \left(\prod_{\alpha\in\Delta_K^+\setminus\Delta_L^+} D_\alpha \right) m_\lambda^{K,L} = \sum_{w\in W_K} \det(w)\varepsilon_{w(\lambda+\delta_K)-\delta_K}$$

Now it is easy to check that the multiplicity function as indicated in figure 7 satisfies this difference equation.

From this multiplicity behaviour it follows immediately that $p_L(Ad(K)\lambda) \cap C_L^+ = supp(M_\lambda^{K,L}) \cap C_L^+$ is the shaded hexagon as indicated in figure 8.

4.5 Futher problems.

In this final section we would like to spend a few words on futher problems. First of all about the relation between the supports of $m_\lambda^{K,L}$ and $M_\lambda^{K,L}$.

<u>Conjecture</u>: $\{supp(m_\lambda^{K,L}) \cap C_L^+\} \subset \{supp(M_\lambda^{K,L}) \cap C_L^+\}$

This conjecture says that, if an irreducible representation $\pi(\mu,L)$ of L occurs in the restriction to L of an irreducible representation $\pi(\lambda,K)$ of K, then the orbit $Ad(L)\mu$ lies in the projection $p_L(Ad(K)\lambda)$. This conjecture, sometimes called the functorial property of the orbit method, was one of the main motivations for our work. The corresponding result for connected nilpotent Lie groups is true and due to Kirillov [18]. L. Auslander and B. Kostant [1] have extended the orbit theory to connected solvable type I Lie groups, and Shchepochkina [23] has shown that in this case the functorial property still holds.

Another problem is to determine explicitly the set $supp(M_\lambda^{K,L}) \cap C_L^+$. In general one can prove (along the same lines as section 3.5) that $p_L(Ad(K)\lambda) \cap C_L^+$ is a polyhedral region. Moreover, it is bounded by portions of $a(w\lambda,W_K^H)$ with $w \in W_K$ and $H \in \mathfrak{t}$ such that the rank of $\Delta_K^H\setminus\Delta_L^H$ is one less than the rank of $\Delta_K \setminus \Delta_L$. As remarked before the inclusion

$$p_L(Ad(K)\lambda) \cap C_L^+ \subset \bigcap_{w\in W_K, w\lambda\in C_L^+} \{w\lambda + \sum_{\alpha\in\Delta_K\setminus\Delta_L, (w\lambda,\alpha)>0} \mathbb{R}^-.\alpha\} \cap C_L^+$$

seems to be quite often an equality. An interesting problem is to find conditions on the pair (K,L) for which the equality holds. Clearly for those pairs the conjecture is true. Even where the equality fails in our examples, we still have that the set $p_L(Ad(K)\lambda) \cap C_L^+$ is a convex polytope. I do not know whether this convexity property is true in general or not.

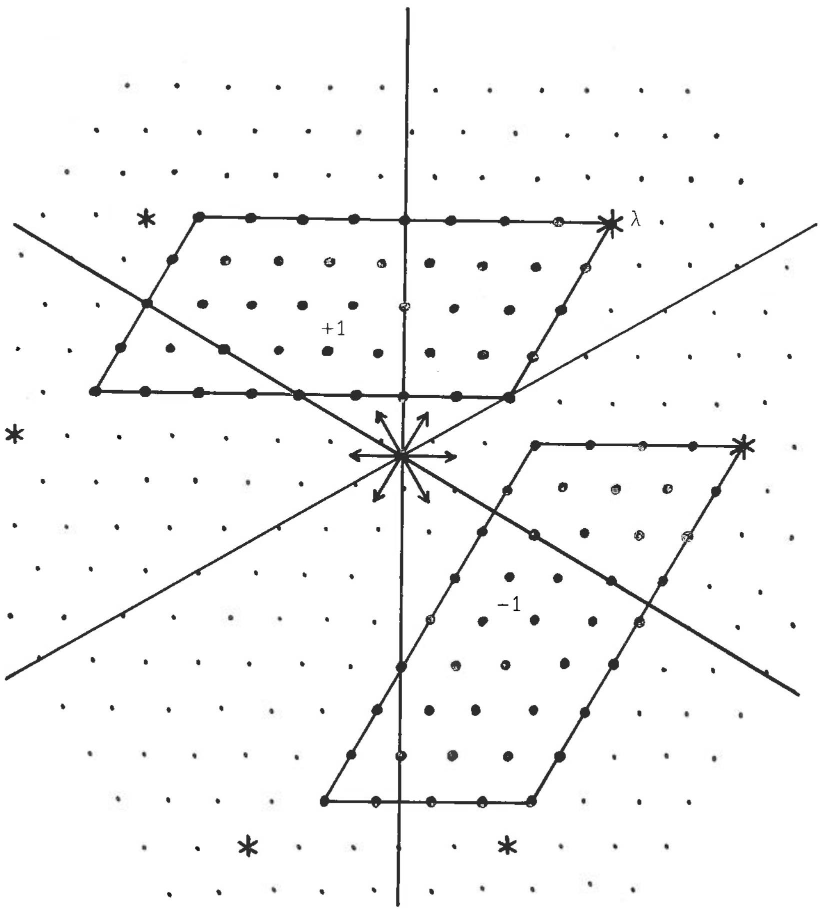


figure 1

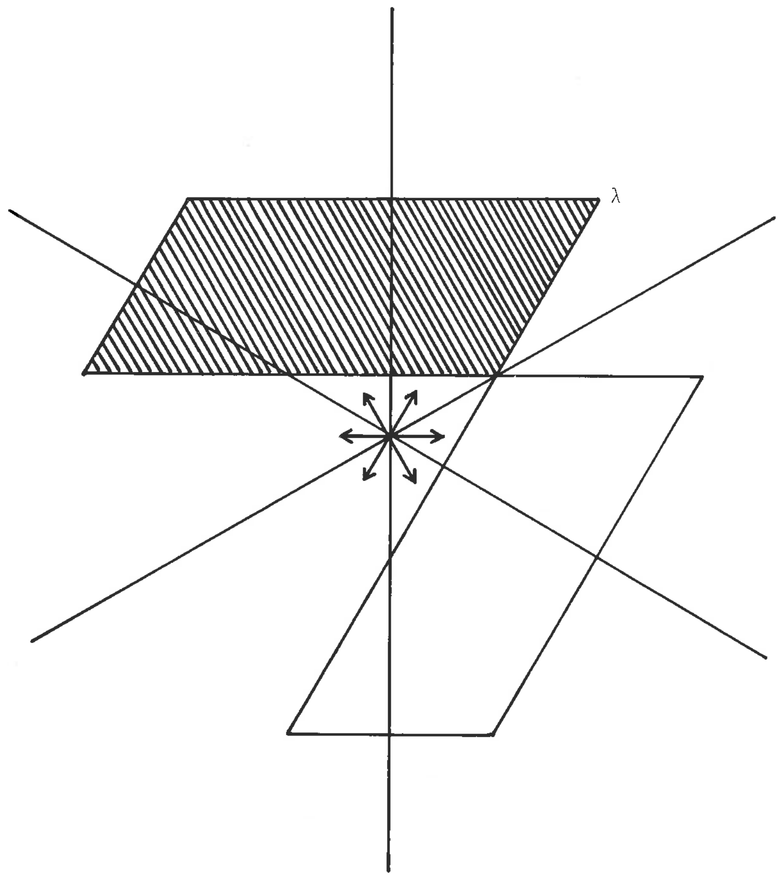


figure 2

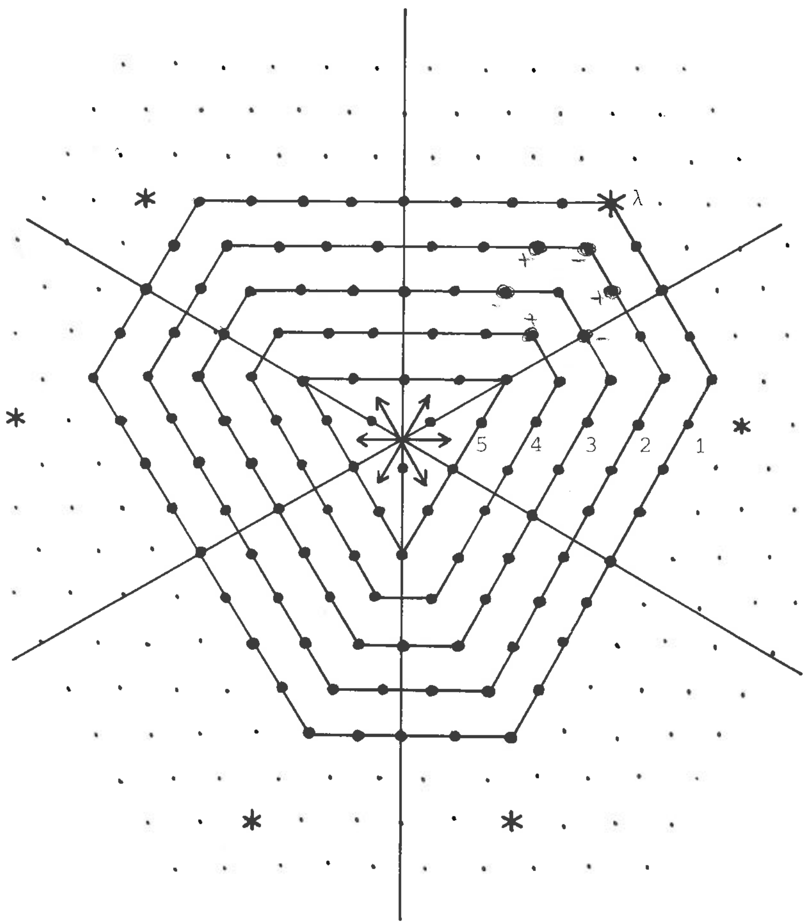


figure 3

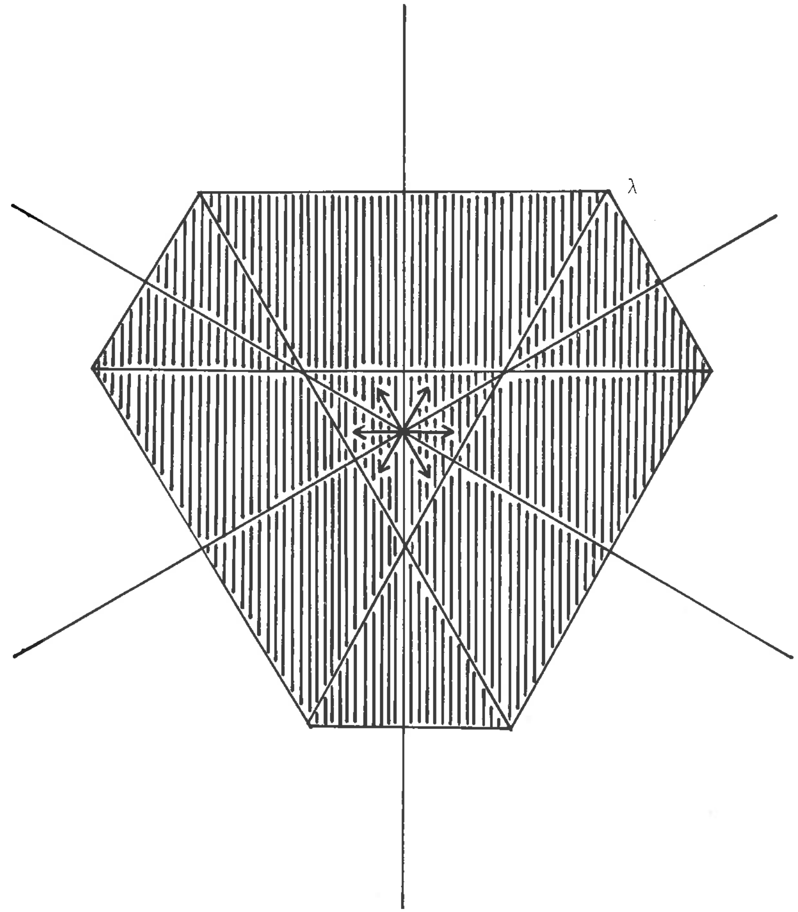


figure 4

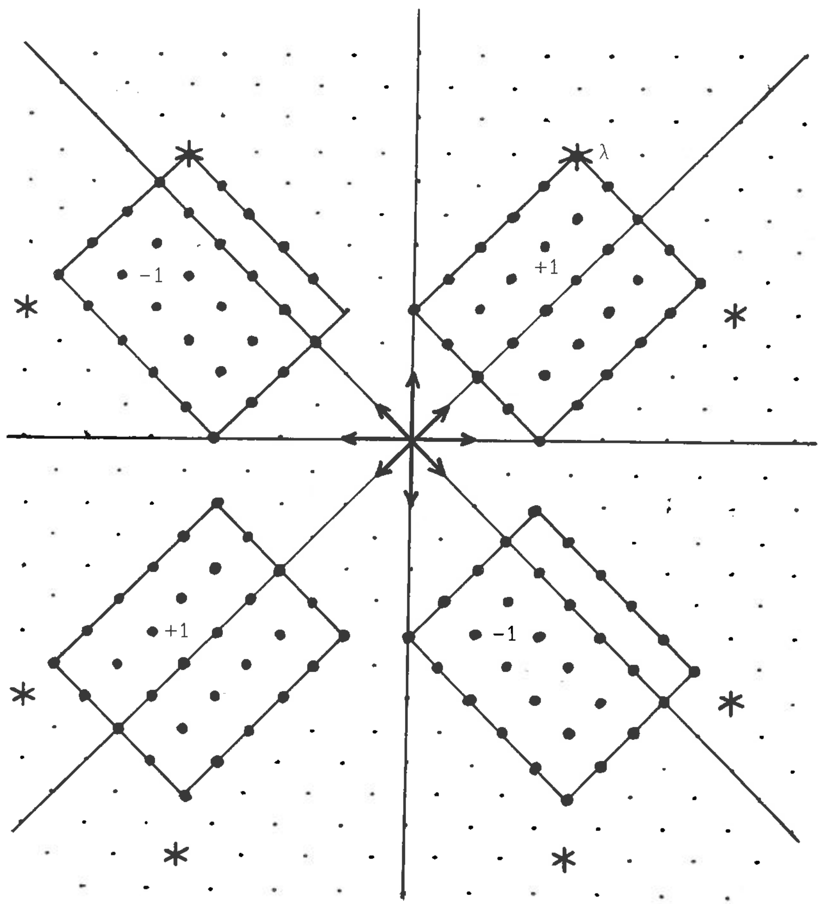


figure 5

SO(5) ↓ SO(4)

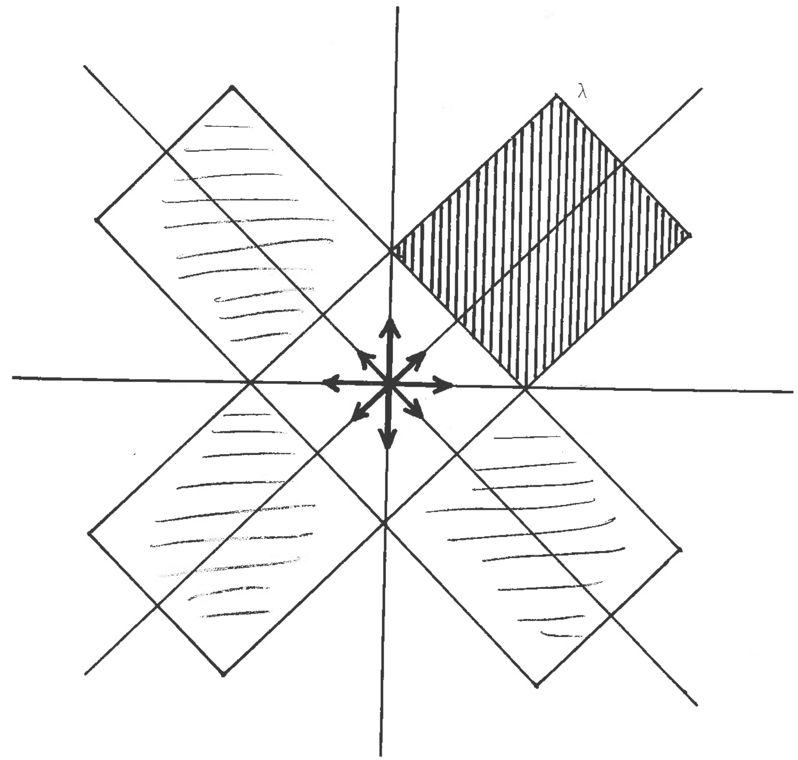


figure 6

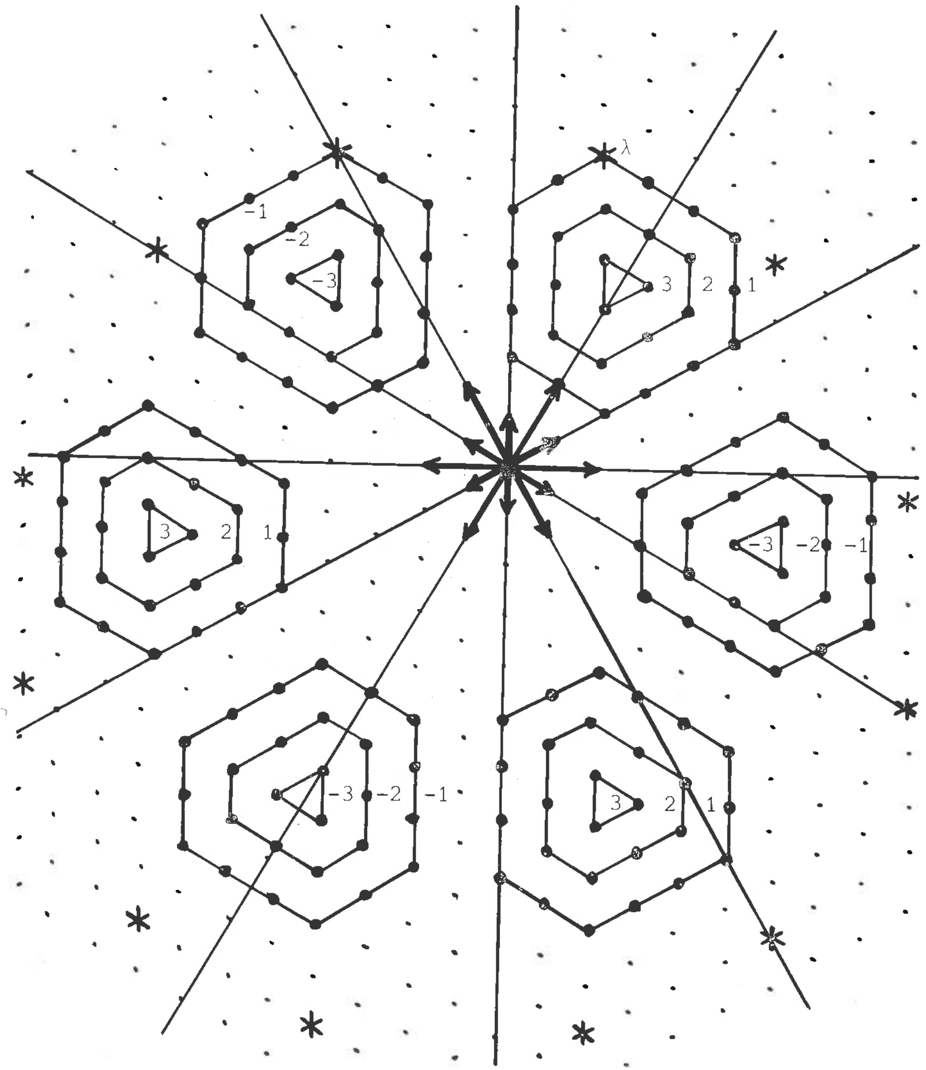


figure 7

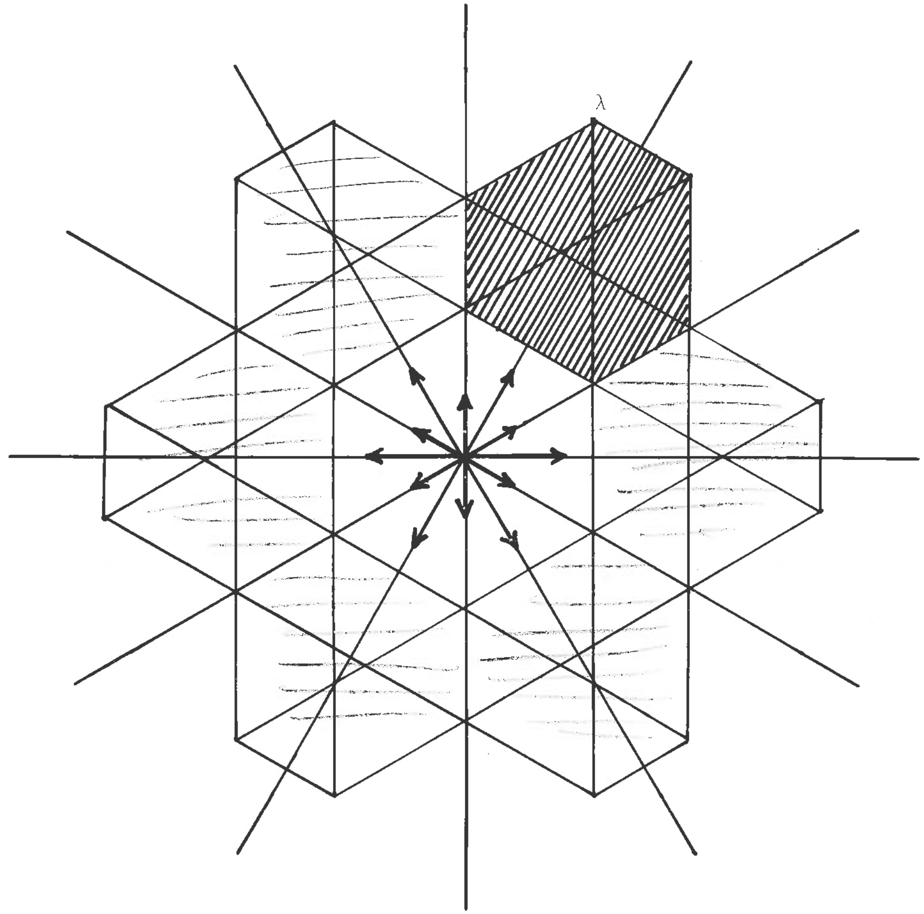


figure 8

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SAMENVATTING

Een van de moeilijkste problemen in de theorie van Liegroepen is om voor een gegeven Liegroep een goede beschrijving te geven van de equivalentieklassen van irreducibele unitaire representaties. Voor samenhangende nilpotente Liegroepen kunnen deze equivalentieklassen op natuurlijke wijze geparametriseerd worden door zekere banen in de duale van de Liealgebra. Deze banenmethode is functorieel, wat inhoudt dat een irreducibele unitaire representatie bij beperking tot een samenhangende gesloten ondergroep zodanig opsplitst als men volgens de projectie van de bijbehorende baan zou verwachten. Voor andere typen Liegroepen laat de representatietheorie zich met wisselend succes beschrijven door dit banenformalisme.

In dit proefschrift wordt bekeken in hoeverre deze functoriele eigenschap doorgaat in het geval van een samenhangende compacte Liegroep bij beperking tot een gesloten ondergroep van dezelfde rang. In de eerste twee hoofdstukken worden projecties van banen bestudeerd los van representatietheorie. In het derde hoofdstuk wordt aangetoond dat de functoriele eigenschap asymptotisch geldt, hetgeen in het vierde hoofdstuk aan de hand van een aantal voorbeelden wordt toegelicht.

CURRICULUM VITAE

De schrijver van dit proefschrift werd geboren op 3 juli 1953 te Lange Ruige Weide. Na het behalen van het einddiploma gymnasium B aan het Comenius college te Hilversum, ving hij in 1971 zijn wiskundestudie te Leiden aan. Hierbij volgde hij colleges van de hoogleraren dr. W.P. Barth, dr. G. van Dijk, dr. A.J.H.M. Van de Ven, dr. C. Visser en van dr. J. Simonis. In 1976 legde hij het doctoraalexamen af.

Sedert 1974 is hij werkzaam bij het Mathematisch Instituut, eerst als studentassistent, en na het doctoraalexamen als wetenschappelijk assistent. In deze laatstgenoemde functie heeft hij onder leiding van Prof. dr. G. van Dijk onderzoek verricht op het gebied van de Liegroepen. Sinds het najaar van 1978 heeft een regelmatig contact met Prof. dr. J.J. Duistermaat hierbij een stimulerende rol gespeeld.

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